Second order absorbing boundary conditions for the wave equation :A solution for the corner problem

A. Bamberger, Patrick Joly, Jean Roberts

UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

Rapports de Recherche

N° 644

SECOND ORDER ABSORBING BOUNDARY CONDITIONS FOR THE WAVE EQUATION: A SOLUTION FOR THE CORNER PROBLEM

Alain BAMBERGER
Patrick JOLY
Jean E. ROBERTS

Mars 1987

SECOND ORDER ABSORBING BOUNDARY CONDITIONS

FOR THE WAVE EQUATION :

A SOLUTION FOR THE CORNER PROBLEM

CONDITIONS LIMITES ABSORBANTES POUR L'EQUATION
DES ONDES : UNE SOLUTION POUR LE PROBLEMEE DU COIN

Alain BAMBERGER[1,2]
Patrick JOLY[2]
Jean. E ROBERTS[2]

[1] Centre de Mathématiques Appliquées, Ecole Polytechnique,
Route de Saclay, 9128 PALAISEAU CEDEX.
INRIA, Domaine de Voluceau, Rocquencourt, 78153 LE CHESNAY.

[2] INRIA, Domaine de Voluceau,
Rocquencourt, 78153 LE CHESNAY.

*This work was supported by a contract between INRIA, the Institut Francaise de Petrole, and the Societe National de ELF AQUITAINE (Petrole)

RESUME

Le traitement des conditions aux limites absorbantes pour l'équation des ondes dans des domaines avec coin est très important d'un point de vue pratique. Une difficulté technique apparait dès que l'on s'intéresse aux conditions d'ordre supérieur ou égal à 2. Nous proposons une solution pour les conditions du second ordre dans le cas bidmensionnel. Cette solution consiste à prescrire une condition au coin spécifique. Le problème obtenu est analysé d'un point de vue théorique. Nous montrons que notre condition est, en un certain sens, optimale. Nos résultats sont illustrés par des simulations numériques. Des extensions aux dimensions supérieures ou aux conditions aux limites d'ordre plus élévé sont proposées.

ABSTRACT

The treatment of domains with corners for the problem of absorbing boundary conditions for the wave equation is very important from a practical point of view. A technical difficulty appears as soon as one considers conditions of order greater than or equal to 2. We propose a solution in the 2D case when second order conditions are used. This solution consists in prescribing an adequate corner condition. The problem thus obtained is analyzed theoretically and our condition is proved to be optimal. Our results are illustrated by numerical simulations. Some extensions to higher space dimensions and higher order conditions are proposed.

MOTS-CLES

Equation des ondes - Conditions aux limites absorbantes - Domaine avec coin.

KEY WORDS

Wave equations - Absorbing boundary conditions - Domain with a corner.

1 - INTRODUCTION

In [4] and [5], Engquist and Majda introduced a sequence of boundary conditions for the 2D-wave equation in the half plane $\{(x_1, x_2), x_2<0\}$:

$$(1.1) \qquad \frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = 0$$

(the velocity is supposed to be constant and taken equal to 1).

These conditions are given recursively by

$$(1.2) \qquad \left|\begin{array}{lll} \mathbf{B}_1 u = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_2} = 0 & & \text{on } \Gamma = \{(x_1,0)\} \\ \mathbf{B}_2 u = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t \partial x_2} - \frac{1}{2}\frac{\partial^2 u}{\partial x_1^2} = 0 & & \text{on } \Gamma \\ \mathbf{B}_{n+1} u = \frac{\partial}{\partial t}(\mathbf{B}_n u) - \frac{1}{4}\frac{\partial^2}{\partial x_1^2}(\mathbf{B}_{n-1} u) = 0 & & \text{on } \Gamma. \end{array}\right.$$

In [5], it is shown that, according to the Kreiss criterion ([6]), each of these boundary conditions is strongly well-posed for the wave equation.

If one assumes that by continuity the interior equation also holds on the boundary one may write

$$\mathbf{B}_n u = \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\right)^n \qquad n = 1,2,\dots,$$

and then it is easy to see that each $\mathbf{B}_n$ is transparent for plane harmonic waves striking the boundary at normal incidence. Further, if u is a plane harmonic wave of unit amplitude striking the boundary at an angle of incidence θ from the normal, then the amplitude R_n of the reflected wave (i.e. the reflection coefficient for $\mathbf{B}_n$) is given by :

$$(1.3) \qquad R_n = \left(\frac{1-\cos\theta}{1+\cos\theta}\right)^n = o(\theta^{2n}), \qquad n = 1,2,\dots .$$

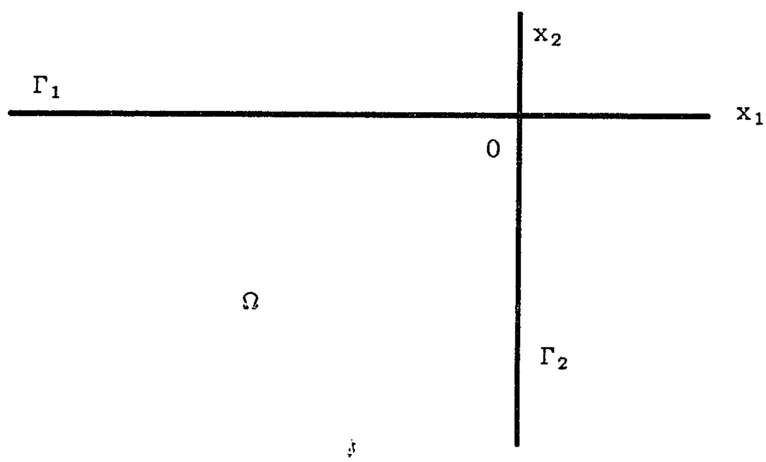
For many practical problems, one is interested in restricting numerical calculations, which should be done in theory on an infinite domain, to a bounded domain.

In the case of a domain with a smooth boundary, a solution was proposed in [5] by Engquist and Majda : this solution generalizes the conditions (1.2) and makes use of the theory of pseudodifferential operators. However, if one uses a finite difference scheme on a uniform grid to compute the numerical solution, it seems more interesting to restrict the effective calculations to a rectangle. For simplicity of exposition, we restrict our attention to the case of the quarter plane $\Omega = \left\{x = (x_1, x_2) \;/\; x_1<0,\; x_2<0\right\}$ and then introduce two absorbing boundaries :

$$\Gamma_1 = \left\{(x_1, 0),\; x_1 \le 0\right\}$$

$$\Gamma_2 = \left\{(0, x_2),\; x_2 \le 0\right\}.$$

We will denote by 0 the corner point $(x_1=0,\; x_2=0)$.



In the case of the first order boundary condition, the initial boundary value problem is written :

$$(1.4)\qquad \left|\begin{array}{ll} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x_2} = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x_1} = 0 & \text{on } \Gamma_2, \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) & \text{in } \Omega. \end{array}\right.$$

This problem is well posed as we have the following energy identy :

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\int_\Omega\left|\frac{\partial u}{\partial t}\right|^2 dx + \frac{1}{2}\int_\Omega|\nabla u|^2\, dx\right) + \int_\Gamma\left|\frac{\partial u}{\partial t}\right|^2 d\sigma = 0. \tag{1.5}$$

($d\sigma$ denotes the superficial measure on Γ.)

Thus the presence of the corner does not pose any specific difficulty. However, as indicated by equation (1.3) and verified by numerical experiments, these first order conditions give rise to significantly larger reflected waves than do those of second or higher order.

If we look at the second order conditions which are written :

$$\left|\begin{array}{ll}\dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial x_2 \partial t} - \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x_1^2} = 0 & \text{on } \Gamma_1, \\[2ex] \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial x_1 \partial t} - \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x_2^2} = 0 & \text{on } \Gamma_2,\end{array}\right. \tag{1.6}$$

it is not at all clear, a priori, whether the corresponding initial boundary value problem is well posed. Numerically, if one uses a classical finite difference scheme (for example see [9], [10]), a naive matching of the conditions (1.5) at 0 is ruled out by the appearance of the second order spatial derivatives in any expression for $\mathbf{B}_2$. Moreover it can be shown, as we shall see in section III, that there is no uniqueness of the finite energy solution for the problem if no condition at the corner is specified. The problem is that it is not clear what corner condition should be chosen. Also, as reported in [5], an improper choice of the corner condition may generate instabilities.

Thus, what we propose to do here is to introduce a boundary condition at the corner 0 and to study this condition from both a mathematical and a numerical point of view.

The outline of this paper is as follows : In section II we explain how we construct our corner condition and present different extensions of the method we use. In section III, we analyse theoretically a family of corner conditions depending on a parameter, containing our condition and also the one proposed previously in [5]. We try to show in what sense our condition is the best one. In section IV, we present various numerical results illustrating the theoretical results of section III.

II - DERIVATION OF THE BOUNDARY CONDITION AT THE CORNER

The principles which guide the construction of our corner condition are quite simple.

(i) we want to eliminate the second order spatial derivatives existing in the boundary conditions on Γ_1 and Γ_2.

(ii) we do not want our corner condition to generate a singularity due to the presence of the corner. In other words we wish to obtain a smooth solution, when the initial data is smooth.

So, let us consider the wave equation (1.1) in Ω with the boundary conditions (1.5) and the initial conditions.

$$(2.1) \qquad \left| \begin{array}{l} u(x,0) = u_0(x) \\ \\ \frac{\partial u}{\partial t}(x,0) = u_1(x) \quad . \end{array} \right.$$

We suppose that

$$(2.2) \qquad \operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \Omega.$$

The idea consists of writing that if u is a $\mathbf{C}^\infty$- solution of (1.1) and (1.5), then by continuity (1.1) and (1.5) should be satisfied at the point 0 for all t :

$$(2.3) \qquad \left| \begin{array}{lll} \frac{\partial^2 u}{\partial t^2} - \Delta u = 0 & 0, & t \geq 0, \\ \\ \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x_2 \partial t} - \frac{1}{2}\ \frac{\partial^2 u}{\partial x_1^2} = 0 & 0, & t \geq 0, \\ \\ \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x_1 \partial t} - \frac{1}{2}\ \frac{\partial^2 u}{\partial x_2^2} = 0 & x_1 = x_2 = 0, & t \geq 0. \end{array} \right.$$

Then we eliminate the spatial second derivatives by adding the two last equations and then substracting one half times the first one from the resulting sum. We obtain

$$\frac{3}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x_1 \partial t} + \frac{\partial^2 u}{\partial x_2 \partial t} = 0, \qquad x=y=0,\ t \geq 0. \tag{2.4}$$

This last equation can be integrated once with respect to time and the corner equation can finally be written

$$\frac{3}{2}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0. \tag{2.5}$$

It is interesting to compare this condition with that proposed by Engquist and Majda in [2],

$$\sqrt{2}\ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0, \tag{2.6}$$

which has the property of being transparent for plane harmonic waves propagating along the diagonal. In practice, conditions (2.5) and (2.6) differ very little though they are obtained from very different considerations. ($\sqrt{2} \approx \frac{3}{2}$). By construction, condition (2.5) could seem superfluous since it appears to be a consequence of the interior and boundary equations and one could wonder whether such a condition is really needed other than for numerical purposes. In fact, the analysis we shall give in section III will prove that it is necessary in the following sense :we can say is that looking for a smooth solution (namely $\mathbf{C}^2$) of (1.1), (1.5), (2.1) is equivalent to looking for a solution of the same problem with the additional corner condition (2.5).

Extension of the corner condition

(i) Generalisation to the three-dimensional case

We consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \qquad t \geq 0,\ x=(x_1,x_2,x_3) \in \Omega, \tag{2.7}$$

for the domain Ω,

$$\Omega = \left\{ x = (x_1,x_2,x_3) \in \mathbb{R}^3;\ x_1<0,\ x_2<0,\ x_3<0 \right\},$$

with initial conditions

$$(2.8)\qquad \begin{aligned} u(x,0) &= u_0(x), && x \in \Omega, \\ \frac{\partial u}{\partial t}(x,0) &= u_1(x), && x \in \Omega, \end{aligned}$$

and second order absorbing boundary conditions

$$(2.9)\qquad \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t \partial x_k} - \frac{1}{2}\left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial x_j^2}\right) = 0, \qquad t>0,\ x \in \Gamma_{ij},$$

for the boundary

$$\Gamma_{ij} = \left\{x \in \mathbb{R}^3;\ x_i<0,\ x_j<0,\ x_k = 0\right\}, \qquad k=1,2,3,$$

and at the corner $0 = (0,0,0)$.

The equation for the edge Δ_k is obtained, exactly as in the 2D case by compatibility between the interior equation and the boundary conditions for the two sides $\Gamma_{\alpha\beta}$ having Δ_k as an edge. We eliminate the second order partial derivatives in the directions i and j orthogonal to Δ_k between these equations and obtain

$$(2.10)\qquad \frac{3}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_i} + \frac{\partial u}{\partial x_j}\right) - \frac{1}{2}\frac{\partial^2 u}{\partial x_k^2} = 0, \qquad t>0,\ x \in \Delta_k.$$

The corner condition can be obtained by eliminating all second order spatial derivatives between the three boundary equations (2.9) and the wave equation, or equivalently between the three edge equations (2.10) and (2.7). The calculations lead, after integration in time, to the condition

$$(2.11)\qquad 2\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3} = 0.$$

Of course, conditions (2.10) and (2.11) can be also generalized to the wave equation in $\mathbb{R}^n$, in the domain Ω :

$$\Omega = \left\{x=(x_1,x_2,\ldots,x_n)\ ;\ \forall\ i \in \{1,2,\ldots,n\},\ x_i < 0\right\}.$$

Let us recall that the second order boundary conditions on the boundary

$$\Gamma_k = \left\{x \in \mathbb{R}^n\ ;\ x_k=0,\ x_j<0 \quad \text{for } j \neq k\right\}$$

can be written

$$(2.12)\qquad \left|\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial t \partial x_k} - \dfrac{1}{2}\ \Delta_k u = 0, \\ \Delta_k u = \sum\limits_{j\neq k} \dfrac{\partial^2 u}{\partial x_j^2} . \end{array}\right.$$

We need then to describe the boundary condition on the edge : (if J is a subset of $\{1,2,...,n\}$)

$$\Delta_J = \left\{x \in \mathbb{R}^n \ /\ x_j = 0 \text{ for } j \in J,\ x_j < 0 \text{ for } j \notin J \right\}.$$

For this, let us define the differential operators

$$\left|\begin{array}{l} \partial_J u = \sum\limits_{j\in J} \dfrac{\partial u}{\partial x_j}, \\ \Delta_J^T u = \sum\limits_{j\notin J} \dfrac{\partial^2 u}{\partial x_j^2} . \end{array}\right.$$

Then, the condition on Δ_J can be written,

$$(2.13)\qquad \frac{k+1}{2}\ \frac{\partial^2 u}{\partial t^2} + \partial_J \frac{\partial u}{\partial t} - \frac{1}{2}\ \Delta_J^T\ u = 0,$$

where k is the number of elements in J.

In particular the corner condition at $0=(0,0,...,0)$ is after time integration : $(J = \{1,2,..,n\})$

$$(2.14)\qquad \frac{n+1}{2}\ \frac{\partial u}{\partial t} + \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} = 0.$$

It is interesting to notice that, using the criterion of Engquist and Majda, the corner condition would be :

$$(2.15)\qquad \sqrt{n}\ \frac{\partial u}{\partial t} + \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} = 0,$$

and that, for sufficiently large n, (2.14) is really quite different from (2.15).

(ii) Generalisation to the third order boundary condition

We come back to dimension 2, and now consider the classical third order boundary conditions (cf. [5]) :

$$(2.16)\qquad \begin{cases} \dfrac{\partial^3 u}{\partial t^3} + \dfrac{\partial^3 u}{\partial x_2 \partial t^2} - \dfrac{3}{4}\dfrac{\partial^3 u}{\partial x_1^2 \partial t} - \dfrac{1}{4}\dfrac{\partial^3 u}{\partial x_1^2 \partial x_2} = 0, & x \in \Gamma_1, \\[2ex] \dfrac{\partial^3 u}{\partial t^3} + \dfrac{\partial^3 u}{\partial x_1 \partial t^2} - \dfrac{3}{4}\dfrac{\partial^3 u}{\partial x_2^2 \partial t} - \dfrac{1}{4}\dfrac{\partial^3 u}{\partial x_1 \partial x_2^2} = 0, & x \in \Gamma_2. \end{cases}$$

Our objective is to obtain, at the point 0, an equation in which none of the second order spatial derivatives in a given direction appear. However, we accept the crossed second derivative $\frac{\partial^2 u}{\partial x_1 \partial x_2}$. For this we first add the equations (2.16) :

$$(2.17)\qquad 2\,\frac{\partial^3 u}{\partial t^3} + \frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x^2}\right) - \frac{3}{2}\frac{\partial \Delta u}{\partial t} - \frac{1}{4}\left(\frac{\partial^3 u}{\partial x_1^2 \partial x_2} + \frac{\partial^3 u}{\partial x_1 \partial x_2^2}\right) = 0.$$

Using the wave equation, we can replace $\frac{\partial \Delta u}{\partial t}$ by $\frac{\partial^3 u}{\partial t^3}$ to obtain

$$(2.18)\qquad \frac{5}{4}\frac{\partial^3 u}{\partial t^3} + \frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}\right) - \frac{1}{4}\left(\frac{\partial^3 u}{\partial x_1^2 \partial x_2} + \frac{\partial^3 u}{\partial x_1 \partial x_2^2}\right) = 0.$$

There remains to eliminate the term $\left(\frac{\partial^3 u}{\partial x_1^2 \partial x_2} + \frac{\partial^3 u}{\partial x_1 \partial x_2^2}\right)$. For this we note that

$$\frac{\partial^3 u}{\partial x_1^2 \partial x_2} + \frac{\partial^3 u}{\partial x_1 \partial x_2^2} = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\Delta u - \left(\frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 u}{\partial x_1^3}\right).$$

Thus, using the wave equation, we arrive at

$$(2.19)\qquad \frac{\partial^3 u}{\partial x_1^2 \partial x_2} + \frac{\partial^3 u}{\partial x_1 \partial x_2^2} = \frac{\partial^3 u}{\partial x_1 \partial t^2} + \frac{\partial^3 u}{\partial x_2 \partial t^2} - \left(\frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 u}{\partial x_2^3}\right).$$

Now, we can differentiate the boundary equation on Γ_j with respect to x_j $(j=1,2)$:

$$(2.20)\qquad \left|\begin{array}{ll} \dfrac{\partial^4 u}{\partial x_1 \partial t^3} + \dfrac{\partial^4 u}{\partial x_1 \partial x_2 \partial t^2} - \dfrac{3}{4}\dfrac{\partial^4 u}{\partial x_1^3 \partial t} - \dfrac{1}{4}\dfrac{\partial^4 u}{\partial x_1^3 \partial x_2} = 0, & x \in \Gamma_1, \\[2ex] \dfrac{\partial^4 u}{\partial x_2 \partial t^3} + \dfrac{\partial^4 u}{\partial x_1 \partial x_2 \partial t^2} - \dfrac{3}{4}\dfrac{\partial^4 u}{\partial x_2^3 \partial t} - \dfrac{1}{4}\dfrac{\partial^4 u}{\partial x_1 \partial x_2^3} = 0, & x \in \Gamma_2. \end{array}\right.$$

We can add these two equations at the point 0 :

$$(2.21)\qquad \frac{\partial^4 u}{\partial x_1 \partial t^3} + \frac{\partial^4 u}{\partial x_2 \partial t^3} + 2\,\frac{\partial^4 u}{\partial x_1 \partial x_2 \partial t^2} - \frac{3}{4}\frac{\partial}{\partial t}\left(\frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 u}{\partial x_2^3}\right) - \frac{1}{4}\frac{\partial^2 \Delta u}{\partial x_1 \partial x_2} = 0.$$

We replace then Δu by $\frac{\partial^2 u}{\partial t^2}$, and note that the resulting equation can be integrated once in time to give

$$(2.22) \qquad \frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 u}{\partial x_2^3} = \frac{4}{3}\left(\frac{\partial^3 u}{\partial x_1 \partial t^2} + \frac{\partial^3 u}{\partial x_2 \partial t^2}\right) + \frac{7}{3}\,\frac{\partial^3 u}{\partial x_1 \partial x_2 \partial t}.$$

Then, using (2.19), we have

$$(2.23) \qquad \frac{\partial^3 u}{\partial x_1^2 \partial x^2} + \frac{\partial^3 u}{\partial x_1 \partial x_2^2} = \frac{7}{3}\left(\frac{\partial^3 u}{\partial x_1 \partial t^2} + \frac{\partial^3 u}{\partial x_2 \partial t} + \frac{\partial^3 u}{\partial x_1 \partial x_2 \partial t}\right).$$

Plugging (2.23) into (2.18) we finally obtain an equation which can still be integrated once in time to lead us finally to our corner condition :

$$(2.24) \qquad 15\,\frac{\partial^2 u}{\partial t^2} + 5\left(\frac{\partial^2 u}{\partial x_1 \partial t} + \frac{\partial^2 u}{\partial x_2 \partial t}\right) - 7\,\frac{\partial^2 u}{\partial x_1 \partial x_2} = 0.$$

Of course, as in the case of the second order boundary condition the corner condition (2.24) can be extended, with no specific difficulty, to the wave equation in $\mathbb{R}^n$. It would also be natural to try to generalize this corner condition to higher order absorbing boundary conditions. This problem does not appear to be so easy. For example, as fourth order spatial derivatives in a given direction appear in the $4^{\underline{th}}$ and $5^{\underline{th}}$ order absorbing boundary conditions (see [5]), our conjecture is that 3 corner conditions are needed and more generally that 2n-1 corner conditions are needed with the $2n^{\underline{th}}$ and $(2n+1)^{\underline{st}}$ absorbing boundary conditions. The problem is now to derive 2n-3 linearly independent equations at the corner, using :

- the interior wave equation (which can be differentiated in time and in any spatial direction),
- each of the two boundary conditions on Γ_j (j=1,2) (which can be differentiated with respect to t and x_j).

This leads to a non trivial algebraic problem. Such a study should be of interest.

We remark that for the $2n^{\underline{th}}$ and $(2n+1)^{\underline{st}}$ conditions where $2n^{\underline{th}}$ order spatial derivatives appear one would expect 2n corner conditions instead of 2n-1, and in fact there is a condition we have implicitly required but have not counted; i.e. that the traces of $u_{|\Gamma_1}$ and of $u_{|\Gamma_2}$ at the corner be the same.

III - MATHEMATICAL ANALYSIS OF THE TWO DIMENSIONAL PROBLEM

In this section, our goal is to study from a mathematical point of view, the following problem :

$$
(P_\gamma)\quad\left|\begin{array}{ll}
\dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, & x \in \Omega,\ t > 0, \\[2ex]
\dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial x_2 \partial t} - \dfrac{1}{2}\ \dfrac{\partial^2 u}{\partial x_2^2} = 0, & x \in \Gamma_1,\ t > 0, \\[2ex]
\dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial x_1 \partial t} - \dfrac{1}{2}\ \dfrac{\partial^2 u}{\partial x_2^2} = 0, & x \in \Gamma_2,\ t > 0, \\[2ex]
\gamma\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x_1} + \dfrac{\partial u}{\partial x_2} = 0, & x = 0,\ t > 0, \\[2ex]
u(x,0) = u_0(x), & x \in \Omega,\ t = 0, \\[2ex]
\dfrac{\partial u}{\partial t}\ (x,0) = u_1(x), & x \in \Omega.
\end{array}\right.
$$

In this problem, we consider a general corner condition :

$$
(\mathrm{CC}\gamma)\qquad \gamma\ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0,
$$

where γ denotes a positive parameter. This generalizes the condition (2.5) we derived in section 2, and the condition (2.6) of Engquist and Majda :

$$
(2.5) \iff \gamma = \gamma^* = \frac{3}{2},
$$

$$
(2.6) \iff \gamma = \sqrt{2}.
$$

III.1 - Weak formulation of the problem

Before investigating existence and uniqueness results, we need to make sense of the problem (P_γ) ; that is we have to make precise the mathematical framework and to define what we shall call a solution of (P_γ). It is clear that it is not difficult to make sense of (P_γ) if the function u is $\mathbf{C}^2$ in $\bar{\Omega}^*$ (with $\bar{\Omega}^*=\bar{\Omega}\backslash\{0\}$) and $\mathbf{C}^1$ in $\Omega \times \mathbb{R}^+$ (with respect to space and time variables). Now, let us assume that there exists a function u such that

$$(3.1)\quad \left|\begin{array}{ll} - & u \in \mathbf{C}^3(\bar{\Omega}^* \times \bar{\mathbb{R}}^+) \cap \mathbf{C}^1(\bar{\Omega}\times[0,\infty)), \\ - & \forall\ t \geq 0,\ u(t) \text{ has compact support}, \\ - & u \text{ satisfies the equations of } (P_\gamma). \end{array}\right.$$

Now, let us differentiate the wave equation with respect to t :

$$\frac{\partial^3 u}{\partial t^3} - \Delta\frac{\partial u}{\partial t} = 0.$$

We multiply (3.1) by a test function $v(x_1, x_2)$ in $\mathbf{C}_0^\infty(\bar{\Omega})$ and integrate in space. We get :

$$\frac{d^3}{dt^3}\left(\int_\Omega uv\ dx\right) - \int_\Omega \Delta\frac{\partial u}{\partial t}\ v\ dx = 0.$$

Using Green's formula, we obtain

$$(3.2)\qquad \frac{d^3}{dt^3}\left(\int_\Omega u(t)\ v\ dx\right) + \frac{d}{dt}\left(\int_\Omega \nabla u(t)\ \nabla v\ dx\right) - \int_\Gamma \frac{\partial^2 u}{\partial n \partial t}\ (t)\ v\ d\sigma = 0.$$

Let us tranform the boundary integral as follows :

$$-\int_\Gamma \frac{\partial^2 u}{\partial n\partial t}\ (t)\ v\ d\sigma = -\int_{\Gamma_1} \frac{\partial^2 u}{\partial x_2 \partial t}\ v\ dx_1 - \int_{\Gamma_2} \frac{\partial^2 u}{\partial x_1 \partial t}\ v\ dx_2.$$

Using the boundary conditions on Γ_1 and Γ_2 we see that

$$(3.3)\qquad -\int_\Gamma \frac{\partial^2 u}{\partial n\partial t}\ v d\sigma = \frac{d^2}{dt^2}\left(\int_{\Gamma_1} uv\ dx_1\right) - \frac{1}{2}\int_{\Gamma_2} \frac{\partial^2 u}{\partial x_1^2}\ v\ dx_1$$
$$+ \frac{d^2}{dt^2}\left(\int_{\Gamma_2} uv\ dx_2\right) - \frac{1}{2}\int_{\Gamma_2} \frac{\partial^2 u}{\partial x_2^2}\ v\ dx_2$$

A double integration by parts leads to the equality

$$(3.4)\quad -\frac{1}{2}\int_{\Gamma_1} \frac{\partial^2 u}{\partial x_1^2}\ v dx_1 - \frac{1}{2}\int_{\Gamma_2} \frac{\partial^2 u}{\partial x_2^2}\ v dx_2 = \frac{1}{2}\int_{\Gamma_1} \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}\ dx_1 + \frac{1}{2}\int_{\Gamma_2} \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\ dx_2$$
$$-\frac{1}{2}\left\{ \frac{\partial u}{\partial x_1}\ (0) + \frac{\partial u}{\partial x_2}\ (0) \right\} v(0)$$

We finally use the corner equation $(CC\gamma)$ to obtain

$$(3.5)\qquad -\frac{1}{2}\int_{\Gamma_1}\frac{\partial^2 u}{\partial x_1^2}\,v\,dx_1-\frac{1}{2}\int_{\Gamma_2}\frac{\partial^2 u}{\partial x_2^2}\,v\,dx_2=\frac{1}{2}\left\{\int_{\Gamma_1}\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1+\int_{\Gamma_2}\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\,dx_2\right\}+\frac{\gamma}{2}\frac{d}{dt}\Big(u(0)\;v(0)\Big).$$

Thus, regrouping equalities (3.2), (3.3) and (3.5), we have proved that u satisfies the following equation :

$$(3.6)\quad\left|\begin{aligned}&\frac{d^3}{dt^3}\Big(\int_\Omega uv\,dx\Big)+\frac{d^2}{dt^2}\Big(\int_\Gamma uv\,d\sigma\Big)\\&+\frac{d}{dt}\Big(\int_\Omega \nabla u\nabla v\,dx+\frac{\gamma}{2}\,u(0)\;v(0)\Big)+\frac{1}{2}\int_\Gamma\frac{\partial u}{\partial\tau}\frac{\partial v}{\partial\tau}\,d\sigma=0,\qquad\forall\, v\in \mathcal{C}_0^\infty(\bar\Omega)\end{aligned}\right.$$

It is clear that (3.6) can have a sense even if u is less regular than required in (3.1). Thus we are led to introduce the functional space

$$(3.7)\quad V=\Big\{\, v\in H^1(\Omega)\ ;\ v_1(x_1)=v(x_1,0)\in H^1(\Gamma_1),\ v_2(x_2)=v(x_2,0)\in H^1(\Gamma_2),\ v_1(0)=v_2(0)\overset{\text{def}}{=}v(0)\Big\}.$$

Note that V is well defined. Indeed, for v in $H^1(\Omega)$ the trace v_1 (resp. v_2) belongs to $H^{1/2}(\Gamma_1)$ (resp. $H^{1/2}(\Gamma_2)$) and thus we can ask if it also belongs to $H^1(\Gamma_1)$ (resp. $H^1(\Gamma_2)$). Moreover, $v_j(0)$ makes sense as v_j belongs to $H^1(]-\infty,0[)$. Note also that V contains $\mathcal{C}_0^\infty(\bar\Omega)$. We define the following norm on V :

$$(3.8)\qquad \|v\|_V^2=\int_\Omega|v|^2\,dx+\int_\Omega|\nabla v|^2\,dx+\int_\Gamma\Big|\frac{\partial v}{\partial\tau}\Big|^2\,d\sigma+|v(0)|^2,$$

($\frac{\partial v}{\partial\tau}$ is the function on $L^2(\Gamma)$ defined by $\frac{\partial v}{\partial\tau}\Big|_{\Gamma_j}=\frac{\partial v_j}{\partial x_j}$, j=1,2).

<u>*THEOREM 3.1*</u>

Equipped with the norm (3.8), V is a Hilbert space.

<u>*Proof*</u> :

Let v_n be a Cauchy sequence in V. Then :

- v_n is a Cauchy sequence in $H^1(\Omega)$,
- $\frac{\partial}{\partial x_j}\left(v_n|\Gamma_j\right)$ is a Cauchy sequence in $L^2(\Gamma_j)$,
- $v_n(0)$ is a Cauchy sequence in $\mathbb{C}$.

Therefore :

- $v_n \longrightarrow v$ in $H^1(\Omega)$,
- $\frac{\partial v_n}{\partial x_j}|\Gamma_j \longrightarrow \dot{v}_j$ in $L^2(\Gamma_j)$,
- $v_n(0) \longrightarrow v_0$ in $\mathbb{C}$.

From the trace theorem in $H^1(\Omega)$, we know that :

$$v_n|\Gamma_j \longrightarrow v|_{\Gamma_j},$$

and then, one easily deduces that

$$\dot{v}_j = \frac{\partial}{\partial x_j}\left(v|\Gamma_j\right).$$

Moreover, as $v_n|\Gamma_j \longrightarrow v|\Gamma_j$ in $H^1(\Gamma_j)$, we know that $v_n(0) \longrightarrow v(0)$ which proves that $v(0) = v_0$ and completes the proof.

□

Remark : Note that V is nothing but the closure of $C_0^\infty(\bar{\Omega})$ for the norm $\|\cdot\|_V$.

To complete our mathematical framework, let us introduce, for H a Hilbert space (with norm $\|\cdot\|_H$) and σ a positive number, the space

$$\text{(3.9)} \qquad L^1_\sigma(\mathbb{R}^+; H) = \left\{v(t) : \mathbb{R}^+ \to H \ / \int_0^{+\infty} \|v(t)\|_H \, e^{-\sigma t} \, dt < +\infty \right\}.$$

The interest of such a space is that it is possible to define the Laplace transform of any v in V by ; (see [2], [3]),

$$(3.10) \qquad \hat{v}(p) = \int_0^{+\infty} v(t)\, e^{-pt}\, dt, \qquad \text{for } Re(p) > \sigma,$$

and to show that the function $p \longrightarrow \hat{v}(p) \in H$ is analytic in the half plane $Re(p) > \sigma$. In particular we have the property

$$(3.11) \qquad \hat{v}(p) = 0 \quad \forall\, p \in \mathbb{R} \,/\, p > \sigma_c \Rightarrow v(t) = 0 \qquad \forall\, t \geq 0.$$

Note also that $L^1_\sigma(\mathbb{R}^+; V) \subset L^1_{loc}(\mathbb{R}^+; V) \subset \mathbf{D}'(\mathbb{R}^+; V)$.

Now, we can give a definition :

Définition 3.1

A function u(t) *is a weak solution of problem* (P_γ) *if and only if there exists* $\sigma > 0$ *such that*

(i) $\quad (u, \frac{du}{dt}) \in L^1_\sigma(\mathbb{R}^+; V) \times L^1_\sigma(\mathbb{R}^+; L^2(\Omega))$;

(ii) $\quad u(0) = u_0 \in V$; $\quad \frac{du}{dt}(0) = u_1 \in L^2(\Omega)$;

(iii)
$$\left| \begin{array}{l} \frac{d^3}{dt^3}\left(\int_\Omega uv dx\right) + \frac{d^2}{dt^2}\left(\int_\Gamma uv d\sigma\right) + \frac{d}{dt}\left(\int_\Omega \nabla u \nabla v\, dx + \frac{\gamma}{2}\, u(0)\, v(0)\right) \\ + \frac{1}{2} \int_\Gamma \frac{\partial u}{\partial \tau} \frac{\partial v}{\partial \tau}\, d\sigma = 0 \qquad \text{in } \mathbf{D}'(\mathbb{R}^+), \ \forall\, v \in V. \end{array} \right.$$

Remarks

• In *(i)*, the derivative $\frac{du}{dt}$ is taken in the sense of distributions with values in $L^2(\Omega)$.

• If one choose $v \in \mathbf{D}(\Omega)$, it is easy to see that *(iii)* can be interpreted as :

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \qquad \text{in } \mathbf{D}'(\mathbb{R}^+ \times \Omega),$$

which shows that, as $\Delta \in \mathcal{L}(H^1(\Omega); H^{-1}(\Omega))$, $\frac{d^2u}{dt^2}$ belongs to the space $L^1_\sigma(\mathbb{R}^+; H^{-1}(\Omega))$. Then we deduce that :

$$u \in \mathbf{C}^1([0,+\infty);\ H^{-1}(\Omega)) \cap \mathbf{C}^0([0,+\infty);\ L^2(\Omega)),$$

which permits us to give a meaning to the initial conditions *(ii)*.

It seems that it is not obvious how to get an existence and uniqueness result with the help of classical methods based on energy identities and a priori estimates. Moreover, our problem does not fall into the category of corner problems previously treated in the literature, [7], [8], [11]. Thus we develop an analysis specific for this problem.

The outline of our analysis will be the following one.

1 We get a general uniqueness result using the Laplace transform (section 3.2).

2 We show the existence of a smooth solution (for smooth data) when $\gamma = \gamma^* = \frac{3}{2}$ (section 3.3).

3 We generalize the existence result to any value of γ by considering the problem satisfied by the difference $u_\gamma - u_{\gamma^*}$ (if u_γ denotes the solution of (P_γ)), pointing out the existence of a "singular" corner wave when $\gamma \neq \gamma^*$. We analyze the properties of this corner wave (section 3.4).

III. 2 - A uniqueness result

Let u be a solution of (P_γ) in the sense of definition (3.1). Then, for $Re(p) > \sigma$, its Laplace transform $\hat{u}(p)$, is at least formally, a solution of

$$(3.12)\quad \begin{cases} -\Delta\hat{u} + p^2\hat{u} = u_1 + p\,u_0 = f & \text{in } \Omega, \\ -\dfrac{1}{2}\dfrac{\partial^2\hat{u}}{\partial x_1^2} + p\dfrac{\partial\hat{u}}{\partial x_2} + p^2\,\hat{u} = 0 & \text{on } \Gamma_1, \\ -\dfrac{1}{2}\dfrac{\partial^2\hat{u}}{\partial x_2^2} + p\dfrac{\partial\hat{u}}{\partial x_1} + p^2\,\hat{u} = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial\hat{u}}{\partial x_1} + \dfrac{\partial\hat{u}}{\partial x_2} + \gamma\,p\,\hat{u} = 0 & \text{at the point } 0, \end{cases}$$

or, more exactly, using the weak formulation, of

$$(3.13)\quad \begin{cases} \hat{u}(p) \in V \\ \hat{a}(p,\hat{u}(p),\hat{v}) = \langle L(p),\hat{v}\rangle \quad , \ \forall\ \hat{v} \in V , \end{cases}$$

where the bilinear form $\hat{a}(p,\cdot,\cdot)$ is defined by

$$(3.14)\quad \begin{aligned} \hat{a}(p,\hat{u},\hat{v}) &= \int_\Omega \nabla\hat{u}\cdot\overline{\nabla\hat{v}}\, dx + p^2 \int_\Omega \hat{u}\,\overline{\hat{v}}\, dx \\ &+ p \int_\Gamma \hat{u}\,\overline{\hat{v}}\, d\sigma + \frac{1}{2p}\int_\Gamma \frac{\partial\hat{u}}{\partial\tau}\,\overline{\frac{\partial\hat{v}}{\partial\tau}}\, d\sigma + \frac{\gamma}{2}\,\hat{u}(0)\,\overline{\hat{v}(0)} , \end{aligned}$$

and the linear form $\langle\hat{L}(p),\cdot\rangle$ is given by

$$(3.15)\quad \langle\hat{L}(p),\hat{v}\rangle = \int_\Omega u_1\,\overline{\hat{v}}\, dx + p\int_\Omega u_0\,\overline{\hat{v}}\, dx.$$

We have the following :

<u>*LEMMA 3.1*</u>

For $\gamma>0$, if p is a positive real number the bilinear form $\hat{a}(p,\hat{u},\hat{v})$ is V elliptic.

<u>*Proof*</u> : It suffices to write

$$\begin{aligned} \hat{a}(p,\hat{u},\hat{v}) &= \int_\Omega |\nabla\hat{u}|^2\, dx + p^2 \int_\Omega |\hat{u}|^2\, dx + p\int_\Gamma |\hat{u}|^2\, d\sigma \\ &+ \frac{1}{2p}\int_\Gamma \left|\frac{\partial\hat{u}}{\partial\tau}\right|^2 d\sigma + \frac{\gamma}{2}\,|\hat{u}(0)|^2 \end{aligned}$$

to deduce that $\hat{a}(p,\hat{u},\hat{u}) \geq \inf\left(1, \frac{1}{p}, p^2, \frac{\gamma}{2}\right) \|\hat{u}\|_V^2$. □

Then, thanks to the Lax-Milgram lemma, problem (3.13) admits, for (u_0, u_1) in $V \times L^2(\Omega)$, a unique solution $\hat{u}(p)$ in H. In particular if $u_0 = u_1 = 0$, we necessarily have $\hat{u}(p) = 0$, for all $p > \sigma$. Using property (3.9), we deduce the following theorem :

THEOREM 3.1

The solution u *of* P_γ, *if it exists, is unique.*

Of course, one could try to develop an existence theory for (P_γ) using the Laplace transform method. One could hope to prove an existence and uniqueness result for problem (3.3) when $p = \eta + i\omega$, $\eta>0$ being fixed, and ω varying in $\mathbb{R}$, and to obtain estimates on $\hat{u}(\eta+i\omega)$ (in V) for any ω in $\mathbb{R}$. We didn't succeed in obtaining such estimates (one can show that such estimates are available in the region $\omega^2 \le 3\eta^2$, but we don't know how to extend these estimates to the whole line $p = \eta + i\omega$).

III.3 - The existence of a smooth solution for $\gamma = \gamma^* = \frac{3}{2}$

In this section, we first assume that the initial data u_0 and u_1 are smooth, namely :

$$(u_0,\ u_1) \in \mathbf{D}(\Omega)\times\mathbf{D}(\Omega), \tag{3.16}$$

and we look for a solution of class $\mathbf{C}^\infty$ of (P_{γ^*}). Of course, it suffices to find a function u satisfying all equations of (P_{γ^*}) except the corner condition. To construct such a function, we first remark that if it exists, then the $\mathbf{C}^\infty$ function v defined by :

$$\begin{aligned} & v = \mathbf{L}_1\ \mathbf{L}_2\ u \\ & \mathbf{L}_1 = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_2 \partial t} - \frac{1}{2}\frac{\partial^2}{\partial x_1^2} \\ & \mathbf{L}_2 = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1 \partial t} - \frac{1}{2}\frac{\partial^2}{\partial x_2^2}, \end{aligned} \tag{3.17}$$

satisfies

$$\begin{aligned} & \frac{\partial^2 v}{\partial t^2} - \Delta v = 0, && x \in \Omega, \quad t > 0, \\ & v_{|\Gamma} = 0, \\ & v(x,0) = v_0(x) && x \in \Omega, \\ & \frac{\partial v}{\partial t}(x,0) = v_1(x), && x \in \Omega, \end{aligned} \tag{3.18}$$

where

$$v_0 = \frac{1}{2}\ \Delta^2 u_0 + \frac{1}{4}\frac{\partial^4 u_0}{\partial x_1^2 \partial x_2^2} + \frac{\partial^2 \Delta u_0}{\partial x_1 \partial x_2} + \frac{1}{2}\ \left(\frac{\partial^3 u_1}{\partial x_1^3} + \frac{\partial^3 u_1}{\partial x_2^3}\right) + \frac{\partial^3 u_1}{\partial x_1^2 \partial x_2} + \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2},$$

(3.19)

$$v_1 = \frac{1}{2}\ \Delta^2 u_1 + \frac{1}{4}\frac{\partial^4 u_1}{\partial x_1^2 \partial x_2^2} + \frac{\partial^2 \Delta u_1}{\partial x_1 \partial x_2} + \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\ \Delta^2 u_0 - \frac{1}{2}\ \left(\frac{\partial^3}{\partial x_1^3} + \frac{\partial^3}{\partial x_2^3}\right)\Delta u_0 .$$

Moreover, we deduce, using the fact that u is a solution of the wave equation, that

$$v = \mathbf{L}u = \frac{1}{4}\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\right)^2\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_2}\right)^2\ u.$$

Now, the process for the construction of a regular solution of (P_{γ^*}) is the following :

(i) We consider the solution v of (3.18) associated to homogeneous Dirichlet boundary conditions.

(ii) We solve, in the quarter plane Ω, the problem :

$$\frac{1}{4}\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\right)^2\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_2}\right)^2\ u = v,\ x \in \Omega \quad ,\ t > 0, \qquad (3.20)_1$$

(3.20)

$$\begin{aligned} &u(x,0) = u_0(x), && x \in \Omega\ , \\ &\frac{\partial u}{\partial t}\ (x,0) = u_1(x), && x \in \Omega\ , \\ &\frac{\partial^2 u}{\partial t^2}\ (x,0) = \Delta u_0(x), && x \in \Omega\ , \\ &\frac{\partial^3 u}{\partial t^3}\ (x,0) = \Delta u_1(x), && x \in \Omega\ . \end{aligned}$$

This problem is well posed since we integrate in the quarter plane $\Omega = \left\{x_1<0,\ x_2<0\right\}$. Moreover u is $\mathbf{C}^\infty$.

THEOREM 3.2

The function u *defined by (3.18), (3.19) and (3.20) is a classical solution of* (P_γ).

Proof : Let us set $w = \frac{\partial^2 u}{\partial t^2} - \Delta u$.

Applying the operator $\Box = \frac{\partial^2}{\partial t^2} - \Delta$ to $(3.20)_1$, we show that :

$$(3.21) \qquad Lw = 0, \qquad \text{in } \Omega,$$

Moreover, it is easy, by simple algebraic calculations, to show that, using (3.19)

$$(3.22) \qquad w(x,0) = \frac{\partial w}{\partial t}(x,0) = \frac{\partial^2 w}{\partial t^2}(x,0) = \frac{\partial^3 w}{\partial t^3}(x,0) = 0, \qquad x \in \Omega.$$

Then (3.21), and (3.22) imply that

$$w = \frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \qquad \text{in } \Omega.$$

It remains to check that the boundary conditions (CL1) and (CL2) hold. For this, let us set

$$w_1 = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\right)^2 u.$$

We have, in light of $(3.20)_1$, that

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_2}\right)^2 w_1 = v, \qquad \text{in } \Omega.$$

In particular, the function, $g_2(x_2,t) = w_1(0,x_2,t)$ satisfies, as w_1 is smooth,

$$(3.23) \qquad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_2}\right)^2 g_2(x_2,t) = v(0,x_2,t) = 0, \qquad x_2 \in \Gamma_2,\ t > 0.$$

Moreover, as the initial data vanish on the boundary Γ, we also have

$$(3.24) \qquad g_2(x_2,0) = \frac{\partial g_2}{\partial t}(x_2,0) = 0.$$

From (3.24), (3.25) we deduce that

$$g_2(x_2,t) = \left\{\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\right)^2 u\right\}(0,x_2,t) = 0, \qquad t>0,\ x_2 \in \Gamma_2. \tag{3.27}$$

But, as u is a solution of the wave equation, we have

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\right)^2 u = \mathbf{L}_2 u, \qquad \text{in } \Omega.$$

In particular, we have, by continuity

$$\mathbf{L}_2 u\big|_{\Gamma_2} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x_1 \partial t} - \frac{1}{2}\ \frac{\partial^2 u}{\partial x_2^2}\Big|_{\Gamma_2} = 0. \tag{3.28}$$

Inverting the roles of x_1 and x_2, .it is easy to see that we also have

$$\mathbf{L}_1 u\big|_{\Gamma_1} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x_2 \partial t} - \frac{1}{2}\ \frac{\partial^2 u}{\partial x_1^2}\Big|_{\Gamma_1} = 0, \tag{3.28}$$

which completes the proof of the theorem. □

It is interesting to analyze the structure of the solution we constructed with the help of the theory of images. Indeed, the solution v of (3.28) can be written as :

$$v = v_I + v_R^1 + v_R^2 + v_R^0,$$

where $(v_I,\ v_R^1,\ v_R^2,\ v_R^0)$ are respectively the solutions in Ω of the solutions of the wave equation in the whole plane $\mathbb{R}^2$ corresponding to the following initial data ($\tilde{v}_0$, $\tilde{v}_1$ being the extensions of v_0, v_1 by 0 outside Ω) :

$$\begin{aligned} v_I(x_1,x_2,0) &= \tilde{v}_0(x_1,x_2) \\ \frac{\partial v_I}{\partial t}(x_1,x_2,0) &= \tilde{v}_1(x_1,x_2); \end{aligned}$$

$$\begin{aligned} v_R^1(x_1,x_2,0) &= -\tilde{v}_0(x_1,-x_2) \\ \frac{\partial v_R^1}{\partial t}(x_1,x_2,0) &= -\tilde{v}_1(x_1,-x_2); \end{aligned}$$

$$v_R^2(x_1,x_2,0) = -\tilde{v}_0(-x_1,x_2)$$

$$\frac{\partial v_R^2}{\partial t}(x_1,x_2,0) = -\tilde{v}_1(-x_1,x_2);$$

$$v_R^0(x_1,x_2,0) = \tilde{v}_0(-x_1,-x_2)$$

$$\frac{\partial v_R^0}{\partial t}(x_1,x_2,0) = \tilde{v}_1(-x_1,-x_2).$$

Then, the solution u constructed in theorem 3.2 can be equivalently decomposed

(3.29) $$u = u_I + u_R^1 + u_R^2 + u_R^0,$$

where u_I is the restriction to Ω of the solution $\tilde{u}_I$ of the wave equation without boundary; i.e., with ($\tilde{u}_0$, $\tilde{u}_1$ being defined as $\tilde{v}_0$, $\tilde{v}_1$),

$$\frac{\partial^2 \tilde{u}_I}{\partial t^2} - \Delta \tilde{u}_I = 0,$$

$$\tilde{u}_I(x,0) = \tilde{u}_0(x),$$

$$\frac{\partial \tilde{u}_I}{\partial t}(x,0) = \tilde{u}_1(x).$$

and where u_R^j (j=1,2,0) denotes the restriction to Ω of the solution $\tilde{u}_R^j$ of :

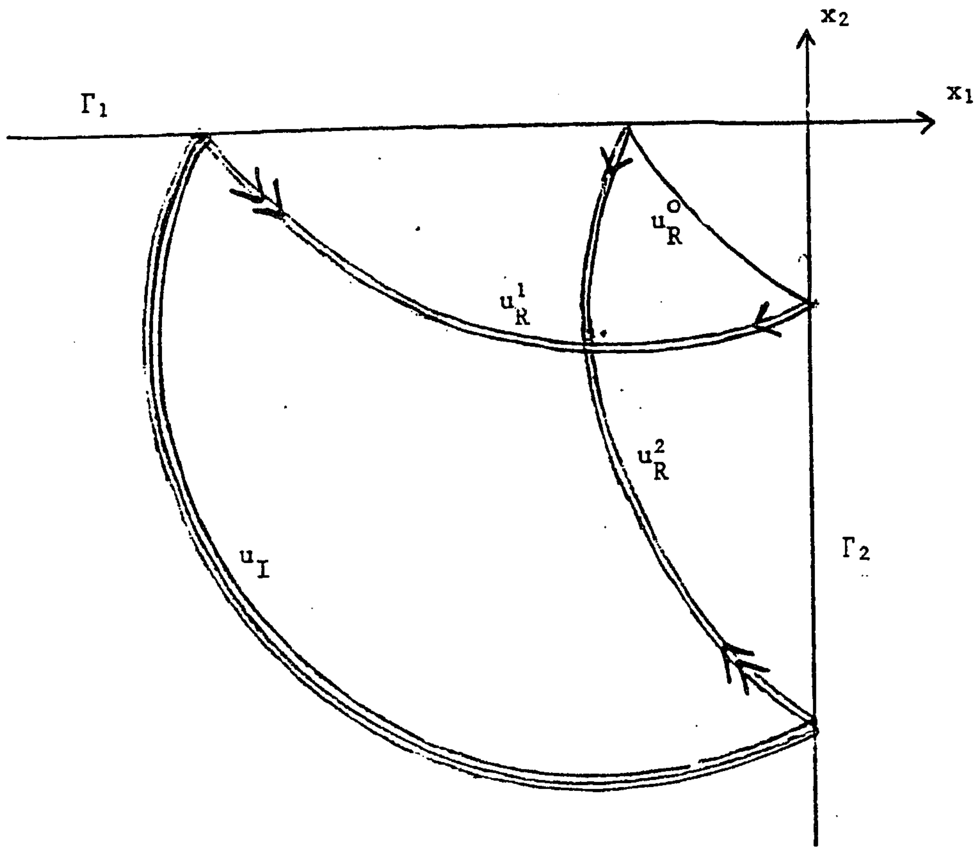
$$\mathbf{L}(\tilde{u}_R^j) = \tilde{v}_R^j,$$

$$\tilde{u}_R^j(x,0) = \frac{\partial \tilde{u}_R^j}{\partial t}(x,0) = \frac{\partial^2 \tilde{u}_R^j}{\partial t^2}(x,0) = \frac{\partial^3 \tilde{u}_R^j}{\partial t^2}(x,0) = 0.$$

The decomposition (3.29) can be interpreted as follows :

- u_I is the incident wave.
- u_R^1 is the wave reflected by the absorbing boundary Γ_1.
- u_R^2 is the wave reflected by the absorbing boundary Γ_2.
- u_R^0 is the secondary reflected wave corresponding to the reflection of u_R^1 on Γ_2 and the one of u_R^2 on Γ_1.

In the case of a point source, we can represent the different wave fronts corresponding to the decomposition (3.29).



The three curves depicting u_I indicate that the amplitude of u_I us greater than that of u_R^1 and u_R^2, each represented by two curves, which is in turn greater than that of u_R represented by a single curve. Also, along u_R^1 and u_R^2 the arrows point in the direction of decreasing amplitude, (cf. numerical results of section IV).

For completeness, we also have to check that the solution u constructed in theorem 3.2 belongs to the class of functions for which we obtain the uniqueness result in section 3.2. For this, let us consider regular functions f and g such that :

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1} = g, \qquad \text{in } \Omega.$$

After multiplication by f and integration over Ω we get : (we set $f_0(x) = f(x,0)$)

$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega |f|^2\, dx\Big) + \int_{\Gamma_2} |f|^2\, dx_2 = \int_\Omega fg\, dx,$$

which implies

$$\frac{1}{2}\int_\Omega |f|^2\, dx = \iint_{0\Gamma_2}^{t} |f|^2\, dx_2\, ds = \frac{1}{2}\int_\Omega |f_0|^2 + \iint_{0\Omega}^{t} fg\, dxds. \tag{3.30}$$

In particular :

$$\frac{1}{2}\int_\Omega |f|^2\, dx \le \frac{1}{2}\int_\Omega |f_0|^2 + \iint_{0\Omega}^{t} fg\, dxds.$$

Then, using Gronwall's lemma, we obtain

$$\int_\Omega |f|^2\, dx \le 2\left\{\int_\Omega |f_0|^2\, dx + \iint_{0\Omega}^{t} |g|^2\, dxds\right\}, \tag{3.31}$$

and, plugging (3.31) into (3.30),

$$\iint_{0\Gamma_2}^{t} |w|^2\, dx_2\, ds \le (1+2t)\int_\Omega |f_0|^2\, dx + \int_0^t (1+t-s)\Big(\int_\Omega |g|^2\, dx\Big)\, ds. \tag{3.32}$$

Now consider two functions w and g define in Ω and related by the sequence of equalities (with $w = w_{1122}$)

$$\frac{\partial w_1}{\partial t} + \frac{\partial w_1}{\partial x_1} = g,$$

$$\frac{\partial w_{12}}{\partial t} + \frac{\partial w_{12}}{\partial x_2} = w_1,$$

$$\frac{\partial w_{112}}{\partial t} + \frac{\partial w_{112}}{\partial x_1} = w_{12},$$

$$\frac{\partial w_{1122}}{\partial t} + \frac{\partial w_{1122}}{\partial x_2} = w_{112}.$$

Then assuming that $\int\limits_{\Omega} |g|^2 \, dx \le \|G_0\|^2$, uniformly in time, we get, using (3.31),

$$\left|\begin{array}{l} \int |w_1|^2 \, dx \le 2 \left(|w_1(0)|^2 + \|G_0\|^2 \, t\right), \\ \int |w_{12}|^2 \, dx \le 2 \left\{|w_{12}(0)|^2 + 2 \, |w_1(0)|^2 \, t + \|G_0\|^2 \frac{t^2}{2}\right\}, \\ \int |w_{112}|^2 \, dx \le 2 \left\{|w_{112}(0)|^2 + 2|w_{12}(0)|^2 \, t + 2 \, |w_1(0)|^2 \, t^2 + \|G_0\|^2 \frac{t^3}{3}\right\}, \end{array}\right.$$

and finally, with the aid of (3.31) and (3.32) we see that

$$(3.33) \quad \left|\begin{array}{l} \int |w|^2 \, dx \le C\left(1+t^4\right)\left(|w(0)|^2 + |w_{112}(0)|^2 + |w_{12}(0)|^2 + |w_1(0)|^2 + \|G_0\|^2\right), \\ \int_0^t\int_{\Gamma_2} |w|^2 dx_2 \, ds \le C\left(1+t^5\right)\left(|w(0)|^2 + |w_{112}(0)|^2 + |w_{12}(0)|^2 + |w_1(0)|^2 + \|G_0\|^2\right). \end{array}\right.$$

We can now invert the roles of x_1 and x_2 in the previous estimates. We then introduce w_2, w_{21}, and w_{221} to obtain

$$(3.34) \quad \left|\begin{array}{l} \int |w|^2 \, dx \le C\left(1+t^4\right)\left(|w(0)|^2 + |w_{221}(0)|^2 + |w_{21}(0)|^2 + |w_2(0)|^2 + \|G_0\|^2\right), \\ \int_0^t\int_{\Gamma_1} |w|^2 \, dx_1 \le C\left(1+t^5\right)\left(|w(0)|^2 + |w_{221}(0)|^2 + |w_{21}(0)|^2 + |w_2(0)|^2 + \|G_0\|^2\right). \end{array}\right.$$

Now, we can apply estimates (3.33) and (3.34) successively to

$$w = \frac{\partial u}{\partial t}, \qquad g = \frac{\partial v}{\partial t};$$

$$w = \frac{\partial u}{\partial x_1}, \qquad g = \frac{\partial v}{\partial x_1};$$

$$w = \frac{\partial u}{\partial x_2}, \qquad g = \frac{\partial w}{\partial x_2};$$

and we can take $\|G_0\|^2 = \int |v_1|^2 \, dx + \int |\nabla v_0|^2 \, dx$ since v satisfies a classical energy identity. Using the relations between (v_0, v_1) and (u_0, u_1) it is easy to get the following estimates :

$$
\left|
\begin{aligned}
&\int \left|\frac{\partial u}{\partial t}\right|^2 dx \le C\left(1+t^5\right)\left(\|u_0\|^2_{H^5(\Omega)} + \|u_1\|^2_{H^4(\Omega)}\right),\\
&\int |\nabla u|^2 dx \le C\left(1+t^5\right)\left(\|u_0\|^2_{H^5(\Omega)} + \|u_1\|^2_{H^4(\Omega)}\right),\\
&\int_\Gamma \left|\frac{\partial u}{\partial t}\right|^2 d\sigma \le C\left(1+t^5\right)\left(\|u_0\|^2_{H^5(\Omega)} + \|u_1\|^2_{H^4(\Omega)}\right),\\
&\int_\Gamma \left|\frac{\partial u}{\partial \tau}\right|^2 d\sigma \le C\left(1+t^5\right)\left(\|u_0\|^2_{H^5(\Omega)} + \|u_1\|^2_{H^4(\Omega)}\right),\\
&\int_\Gamma \left|\frac{\partial u}{\partial \tau}\right|^2 d\sigma \le C\left(1+t^5\right)\left(\|u_0\|^2_{H^5(\Omega)} + \|u_1\|^2_{H^4(\Omega)}\right),
\end{aligned}
\right.
$$

and then to deduce that the solution u we constructed in theorem 3.2 is a weak solution in the sense of definition 3.1. Moreover, these estimates permit us to generalize the corresponding existence and uniqueness result to the case of initial data (u_0,u_1) in $H^5(\Omega) \times H^4(\Omega)$ (this result is probably not optimal though we have not succeeded in improving it).

III.4 - THE CASE $\gamma \neq \frac{3}{2}$: EXISTENCE OF A CORNER WAVE

We return now to the problem P_γ for arbitrary $\gamma>0$. We again assume that all initial data is C^∞ and is compactly supported in Ω. First suppose that u_γ is a solution to P_γ for some given γ. If we denote by u the regular solution of P_γ for $\gamma=\frac{3}{2}$ then we may define

$$v_\gamma = u_\gamma - u^*.$$

Thus we can decompose u_γ

$$u_\gamma = u^* + v_\gamma,$$

where u^* is regular everywhere and v_γ is a function which clearly satisfies the following equations : (we omit here the index γ for simplicity)

$$
\tilde{P}_\gamma \quad
\left|
\begin{array}{lll}
\dfrac{\partial^2 v}{\partial t^2} - \Delta v = 0, & x \in \Omega,\ t > 0, & \\[2ex]
\dfrac{\partial^2 v}{\partial t^2} + \dfrac{\partial^2 v}{\partial t \partial x_2} - \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x_1^2} = 0, & x \in \Gamma_1,\ t > 0, & \text{CL1}\\[2ex]
\dfrac{\partial^2 v}{\partial t^2} + \dfrac{\partial^2 v}{\partial t \partial x_1} - \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x_2^2} = 0, & x \in \Gamma_2,\ t > 0, & \text{CL2}\\[2ex]
\dfrac{3}{2}\dfrac{\partial^2 v}{\partial t^2} + \dfrac{\partial}{\partial t}\left(\dfrac{\partial v}{\partial x_1} + \dfrac{\partial v}{\partial x_2}\right) = g, & x = 0,\ t > 0, & (\text{CC}\gamma)'\\[2ex]
v(x,0) = \dfrac{\partial v}{\partial t}(x,0) = 0, & x \in \Omega, &
\end{array}
\right.
$$

with

$$g = g_\gamma = (\gamma - \gamma^*) \frac{\partial^2 u^*}{\partial t^2}. \tag{3.35}$$

(We remark that in this section for simplicity of exposition we have chosen to work with the corner condition $(CC\gamma)'$ instead of $(CC\gamma)$ since with the zero initial conditions $(CC\gamma)'$ and $(CC\gamma)$ are equivalent).

What we shall do in the following is to show that there exists a solution H_γ of $\tilde{P}_\gamma$ with $g = g_\gamma$. We shall show that v_γ has a singularity at the corner and shall say that v_γ is the corner wave due to the corner condition $(CC\gamma)$. Finally the solution u_γ to P_γ shall be obtained by adding the regular solution u^* of P_{γ^*} to v_γ ; $u_\gamma = v_\gamma + u^*$.

To construct the solution H_γ to $\tilde{P}_\gamma$ we shall proceed much as for the construction in III.3 of the regular solution u^* to P_{γ^*} except here we need to introduce a singularity at the corner point. Thus we shall start with the elementary solution G to the wave equation in $\mathbb{R}^2$ at the point (0,0). Then, as before, we shall integrate G along the caracteristics of $\mathbf{L}$ ((3.20)) to obtain $\tilde{G}$ which should satisfy the wave equation as well as the zero initial conditions of $\tilde{P}_\gamma$. However, as G does not vanish on the boundary Γ, there is no reason why $\tilde{G}$ should satisfy the boundary conditions CL1 and CL2 of $\tilde{P}_\gamma$. But, we observe that $\frac{\partial^2 G}{\partial x_1 \partial x_2}$, which will play the same role here as does v in III.3, does vanish on Γ; so taking the corresponding derivative of $\tilde{G}$, $H = \frac{\partial^2 \tilde{G}}{\partial x_1 \partial x_2}$, we shall obtain a function H which we shall demonstrate satisfies all of the equations of $\tilde{P}_\gamma$ except the corner condition. Moreover, we shall show that a constant multiple $H_\gamma = a_\gamma H$ of H does indeed satisfy the condition $(CC\gamma)'$.

To be more precise, we let G denote the solution to

$$\frac{\partial^2 G}{\partial t^2} - \Delta G = \delta(x) \times \delta(t), \qquad x \in \mathbb{R}^2,\ t > 0,$$

$$G(x,0) = \frac{\partial G}{\partial t}(x,0) = 0, \qquad x \in \mathbb{R}^2.$$

Then G is given explicitly by

$$G(x,t) = \begin{cases} \frac{1}{2}\,\pi(t^2-x_1^2-x_2^2)^{-\frac{1}{2}}\,, & x_1^2+x_2^2 < t^2, \\ 0\,, & \text{otherwise.} \end{cases}$$

Next we integrate G twice in the direction $(t+x_1)$ and twice in the direction $(t+x_2)$ to obtain the solution $\tilde{G}$ to the problem :

$$\text{(3.36)} \quad \begin{cases} L\tilde{G} = \frac{1}{4}\left(\frac{\partial}{\partial t} \quad \frac{\partial}{\partial x_1}\right)^2 \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\right)^2 \tilde{G} = g, & x \in \mathbb{R}^2,\ t > 0, \\ \tilde{G}(x,0) = \frac{\partial \tilde{G}}{\partial t}(x,0) = \frac{\partial^2 \tilde{G}}{\partial t^2}(x,0) = \frac{\partial^3 \tilde{G}}{\partial t^3}(x,0) = 0, & x \in \mathbb{R}^2. \end{cases}$$

An explicit expression for $\tilde{G}$ may be obtained by straight forward calculation :

$$\text{(3.37)} \quad \tilde{G} = \begin{cases} \frac{1}{12}\,(t-x_1-x_2)^3\, F(b(x,t)), & x_1^2+x_2^2 < t^2, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\text{(3.38)} \quad \begin{cases} F(b) = \int_0^b \int_0^s \frac{\sigma^{\frac{3}{2}}}{(1+\sigma)^2}\, d\sigma\, ds, \\ b(x,t) = (t^2-x_1^2-x_2^2) \,/\, (t-x_1-x_2)^2. \end{cases}$$

Remark : One may further calculate that

$$\text{(3.39)} \quad F(b) = (5 + \frac{4}{3}\, b)\sqrt{b} - (5+3b) \arctan\sqrt{b},$$

with b given by (3.38); though, as we are interested in the derivatives of $\tilde{G}$, the form (3.38) is easier to work with.

Concerning the regularity of $\tilde{G}$, we note that $\tilde{G}$ is bounded but singular along the plane $x_1+x_2=t$. However, the intersection of this plane with the domain that interests us, $\bar{\Omega} \times \mathbb{R}^+$, is the point $x=0$, $t=0$.

To study the regularity of the derivatives of $\tilde{G}$ we define Q to be the intersection of the cone $x_1^2+x_2^2<t^2$, $t>0$, with the octant $x_1\le 0$, $x_2\le 0$, $t>0$, i.e. with $\bar{\Omega} \times \mathbb{R}^+$, and denote by K the intersection of the boundary of Q with the boundary of the cone $x_1^2+x_2^2<t^2$, i.e. $K = \bar{Q} \setminus Q$.

<u>*THEOREM 3.3*</u> :

i) $\tilde{G}$ vanishes in $(\bar{\Omega} \times \mathbb{R}^+_*) \setminus \bar{Q}$.

ii) $\tilde{G}$ is C^∞ in Q.

iii) $\tilde{G}$ is C^3 in $(\bar{\Omega} \setminus \{0\}) \times \mathbb{R}^+$.

iv) The fourth derivatives of $\tilde{G}$ have singularities of the form $(t^2-x_1^2-x_2^2)^{-\frac{1}{2}}$ along K.

v) $\tilde{G}$ satisfies the wave equation

$$\frac{\partial^2 \tilde{G}}{\partial t^2} - \Delta\tilde{G} = 0, \qquad x \in \Omega,\ t > 0,$$

with initial conditions

$$\tilde{G}(x,0) = \frac{\partial}{\partial t}\tilde{G}(x,0) = 0, \qquad x \in \Omega.$$

<u>*Proof*</u> : By (3.37) $\tilde{G}$ vanishes outside the cone $x_1^2+x_2^2 \le t^2$, thus in $\bar{\Omega} \setminus \bar{Q}$.

As b is C^∞ in $\bar{Q} \setminus \{0\}$, to study the regularity of $\tilde{G}$ in $\bar{Q} \setminus \{0\}$, it suffices to consider the regularity of F. F is C^∞ for $b>0$; it is C^3 for $b \ge 0$, its fourth derivative having a singularity at $b=0$ of the form $b^{-\frac{1}{2}}$. Since the zeros of b are the points of the boundary of the cone $t^2 \le x_1^2+x_2^2$ whose intersection with $\bar{Q}$ is K, *ii)*, *iii)*, and *iv)* follow.

The proof that $\tilde{G}$ satisfies the wave equation is essentially the same as that that the function u of section III.3 satisfies the wave equation ... We note that the function R(x,t) defined by $R = \frac{\partial^2}{\partial t^2}\tilde{G} - \Delta\tilde{G}$ satisifes

$$\mathbb{L}R = 0, \qquad x \in \mathbb{R}^2,\ t > 0,$$

and it clearly also satisfies the zero initial conditions

$$R(x,0) = \frac{\partial R}{\partial t}(x,0) = \frac{\partial^2 R}{\partial t^2}(x,0) = \frac{\partial^3 R}{\partial t^3}(x,0) = 0, \qquad x \in \mathbb{R}^2.$$

Thus one concludes by uniqueness that $R=0$. The demonstration of the theorem is completed on remarking that by its definition $\tilde{G}$ satisfies the zero initial conditions of $\tilde{P}_\gamma$.

Next we introduce the function H defined by

$$H(x,t) = \frac{\partial^2 \tilde{G}}{\partial x_1 \partial x_2}(x,t).$$

The following result is an immediate consequence of the preceeding theorem.

<u>*COROLLARY 3.1*</u>

i) H *vanishes in* $\bar{\Omega} \setminus \bar{Q}$

ii) H *is* C^∞ *in* Q.

iii) H *is* C^1 *in* $\bar{\Omega} \setminus \{0\}$.

iv) The second derivatives of H *are singular along* K.

v) H *satisfies the wave equation in the distributional sense in* $\mathbb{R}^2 \times \mathbb{R}^+_*$ *and in the classical sense in* $\bar{\Omega} \times \mathbb{R}^+_*$ *except along* K; *in particular :*

$$\frac{\partial^2 H}{\partial t^2} - \Delta H = 0, \qquad (x,t) \in Q.$$

H *also satisfies the initial conditions*

$$H(x,0) = \frac{\partial H}{\partial t}(x,0) = 0, \qquad x \in \Omega.$$

We need now to show that H satisfies the boundary conditions of $\tilde{P}_\gamma$.

<u>*LEMMA 3.2*</u> :

H *satisfies the boundary conditions* CL1 *and* CL2.

<u>*Proof*</u> *:* To see that CL1 holds for H along Γ_1 we consider first the part of Γ_1 interior to the cone $x_1^2+x_2^2<t^2$ or more precisely $(\Gamma_1 \times \mathbb{R}^+) \cap Q$. Since H satisifes the wave equation in the classical sense in the interior of the cone, it suffices to show that

$$\tilde{\mathbf{L}}_2\ H = \frac{1}{2}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_2}\right)^2 H = 0.$$

We know that $\tilde{\mathbf{L}}_2\ \tilde{G}$ is the solution $\tilde{G}_1$ to

$$\tilde{\mathbf{L}}_1\ \tilde{G}_1 = \frac{1}{2}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\right)^2 \tilde{G}_1 = G, \qquad x \in \mathbb{R}^2,\ t > 0,$$

$$\tilde{G}_1(x,0) = \frac{\partial}{\partial t}\ \tilde{G}_1(x,0) = 0, \qquad x \in \mathbb{R}^2.$$

One easily calculates that

$$\tilde{G}_1 = \frac{1}{(t-x_1)^2}\ \frac{1}{3}\ (t^2-x_1^2-x_2^2)^{\frac{3}{2}},$$

and that

$$\tilde{\mathbf{L}}_2\ H = \frac{\partial^2}{\partial x_1 \partial x_2}\ \tilde{G}_1 = \frac{-2x_2}{(t-x_1)^3}\ (t^2-x_1^2-x_2^2)^{\frac{1}{2}} + \frac{x_1\ x_2}{(t-x_1)^2}\ (t^2-x_1^2-x_2^2), \tag{3.40}$$

which vanishes for $x_2 = 0$.

Clearly CL1 holds for H on the part of $\Gamma_1 \times \mathbb{R}^+$ outside the cone $x_1^2+x_2^2 \leq t$ as H vanishes there.

We shall show that CL1 holds along Γ_1 in the sense that $\mathbf{L}_2$ H is integrable along Γ_1 with L^1 norm equal to 0. Toward this end we remark that since the wave equation holds throughout Q that (3.37) gives an expression for $\mathcal{L}_2 H$ there, i.e.

$$\mathbf{L}_2\ H = \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_2 \partial t} - \frac{1}{2}\ \frac{\partial^2}{\partial x_1^2}\right) H$$

$$= \frac{-2x_2}{(t-x_1)^3}\ (t^2-x_1^2-x_2^2)^{\frac{1}{2}} + \frac{x_1\ x_2}{(t-x_1)^2}\ (t^2-x_1^2-x_2^2)^{-\frac{1}{2}}, \qquad (x,t) \in Q.$$

Thus we may define a function

$$H_2(x_1,x_2,t) = \begin{cases} \mathbf{L}_2\ H(x_1,x_2,t) & (x,t) \in Q \\ 0 & \text{otherwise.} \end{cases}$$

Then H_2 has a singularity along the boundary of the cone $x_1^2+x_2^2 \leq t$ of the form $1/\sqrt{t^2-x_1^2-x_2^2}$. Define the function H_2^ϵ for $\epsilon < 0$ by

$$H_2^\epsilon(x_1,t) = H_2(x_1,\epsilon,t).$$

Then H_2^ϵ is an L^1 function and one can calculate that H_2^ϵ tends toward 0 in the L^1 norm as ϵ tends to 0. (That CL1 is satisfied along Γ_1 in this sense implies that it is satisfied in the weak sense of definition 3.1.)

It follows from the symmetry of H with respect to x_1 and x_2 that condition CL2 is satisfied on Γ_2.

Now we have that H satisfies all of the conditions of $\tilde{P}_\gamma$ except the corner condition. Thus we want to investigate the behavior of H at the corner $x_1=x_2=0$. As H is C^∞ in a neighborhood of the corner when $t>0$, the following lemma is immediate.

LEMMA 3.3 : *H satisfies $(CC\gamma_*)'$ with $g=0$ for $t>0$.*

We can also show the following lemma concerning the behavior of H at the corner for $t>0$.

LEMMA 3.4 : *The first derivatives of H in time and in space at the corner $x_1 = x_2 = 0$, for $t>0$, are constant in time and positive, i.e.*

(3.40)
$$\frac{\partial H}{\partial t}(0,t) = c_1 > 0, \qquad t > 0,$$
$$\frac{\partial H}{\partial x_1}(0,t) = \frac{\partial H}{\partial x_2}(0,t) = c_2 > 0, \qquad t > 0.$$

Proof : We consider first the time derivative. From (3.37), (3.38) we calculate

(3.41)
$$H = \frac{\partial^2}{\partial x_1 \partial x_2}\tilde{G} = \frac{1}{2}(t-x_1-x_2)\,F(b) - \frac{1}{4}(t-x_1-x_2)^2\,F'(b)\left(\frac{\partial b}{\partial x_1} + \frac{\partial b}{\partial x_2}\right)$$
$$+ \frac{1}{12}(t-x_1-x_2)^3\left(F''(b)\,\frac{\partial b}{\partial x_1}\,\frac{\partial b}{\partial x_2} + F'(b)\,\frac{\partial^2 b}{\partial x_1 \partial x_2}\right).$$

Now for $x_1=x_2=0$ and $t>0$, $b=1$ while $\dfrac{\partial b}{\partial x_1} = \dfrac{\partial b}{\partial x_2} = \dfrac{2}{t}$ and $\dfrac{\partial^2 b}{\partial x_1 \partial x_2} = \dfrac{6}{t^2}$. Thus

$$H(0,t) = \frac{t}{4}\,(2F(1) - 2F'(1) + \frac{4}{3}\,F''(1)), \qquad t>0,$$

and we conclude that $\frac{\partial H}{\partial t}$ (0,t) is constant, t>0, In fact we can calculate the value of the constant :

$$F(b) - F'(b) = \int_0^b \frac{(b-\sigma)\sigma^{\frac{3}{2}}}{(1+\sigma)^2}\,d\sigma - \int_0^b \frac{\sigma^{\frac{3}{2}}}{(1+\sigma)^2}\,d\sigma$$
$$= \int_0^b \frac{(b-\sigma-1)\sigma^{\frac{3}{2}}}{(1+\sigma)^2}\,d\sigma,$$

and

$$F(1) - F'(1) = -\int_0^1 \frac{\sigma^{\frac{5}{2}}}{(1+\sigma)^2}\,d\sigma,$$

which one can calculate explicitly to be $\frac{23}{6} - \frac{5\pi}{4}$. As $F''(1) = \frac{1}{4}$ we have

$$\frac{\partial H}{\partial t}\,(0,t) = 2 - \frac{5\pi}{8} > 0.$$

As H is symmetric in the variables x_1, x_2 we clearly have $\frac{\partial H}{\partial x_1} = \frac{\partial H}{\partial x_2}$ at x=0. We can obtain an expression for $\frac{\partial H}{\partial x_1}$ from (3.37) by straightforward calculation which permits us to determine that $\frac{\partial H}{\partial x_1}$ is constant in time at $x_1=x_2=0$, t>0, but which is not very ameanable for determining that the constant is positive. Thus it is simpler to proceed in an indirect fashion ...

As H satisfies the wave equation in Ω, t>0, in the distributional sense, on integrating over Ω and differentiating in time, we obtain

$$\frac{d^3}{dt^3}\int_\Omega H\,dx_1\,dx_2 - \frac{d}{dt}\int_\Gamma \frac{\partial H}{\partial \nu}\,d\gamma = 0, \qquad t>0.$$

Integrating the conditions CL1 over Γ_1 and CL2 over Γ_2 and adding we have

$$\frac{d^2}{dt^2}\int_\Gamma H\,d\gamma + \frac{d}{dt}\int_\Gamma \frac{\partial H}{\partial \nu}\,d\gamma = \frac{1}{2}\,\Big[\frac{\partial H}{\partial x_1}\,(0,0,t) + \frac{\partial H}{\partial x_2}\,(0,0,t)\Big].$$

We then combine these two equations to arrive at

$$\frac{d^3}{dt^3}\int_\Omega H\,dx_1\,dx_2 + \frac{d^2}{dt^2}\int_\Gamma H\,d\gamma = \frac{1}{2}\,\Big[\frac{\partial H}{\partial x_1}\,(0,0,t) + \frac{\partial H}{\partial x_2}\,(0,0,t)\Big].$$

The relation between H and $\tilde{G}$, $H = \dfrac{\partial^2 \tilde{G}}{\partial x_1 \partial x_2}$, gives us :

$$\frac{\partial^3}{\partial t^3} \tilde{G}(0,0,t) + \frac{\partial^2}{\partial t^2} \left[\frac{\partial \tilde{G}}{\partial x_1} (0,0,t) + \frac{\partial \tilde{G}}{\partial x_2} (0,0,t)\right] = \frac{1}{2} \left[\frac{\partial H}{\partial x_1} (0,0,t) + \frac{\partial H}{\partial x_2} (0,0,t)\right]$$

One calculates from (3.37), (3.38)

$$\tilde{G}(0,0,t) = \frac{1}{12} t^3 F(1)$$

$$\frac{\partial \tilde{G}}{\partial x_1} (0,0,t) = \frac{\partial \tilde{G}}{\partial x_2} (0,0,t) = \frac{t^2}{4} \left[-F(1) + \frac{2}{3} F'(1)\right].$$

Thus we have

$$\begin{aligned} \frac{\partial H}{\partial x_1} (0,0,t) = \frac{\partial H}{\partial x_2} (0,0,t) &= - \frac{1}{2} F(1) + \frac{2}{3} F'(1) \\ &= \frac{1}{2} \int_0^1 \frac{(\sigma+1/3)\sigma^{\frac{3}{2}}}{(1+\theta)^2} d\sigma \\ &= \frac{\pi-3}{2} > 0 \end{aligned}$$

□

COROLLARY 3.2 : *For* $\gamma>0$, *we have along the axis* $(0,0,t)$, $t>0$,

$$\frac{d}{dt} \left(\gamma \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x_1} + \frac{\partial H}{\partial x_2}\right) = \frac{1}{a_\gamma} \delta(0)$$

where the amplitude a_γ *is defined by*

(3.42) $$\frac{1}{a_\gamma} = \gamma c_1 + 2c_2 > 0.$$

Thus we may define

(3.43) $$H_\gamma = a_\gamma H,$$

and it follows that

$$\frac{d}{dt} \left(\gamma \frac{\partial H_\gamma}{\partial t} + \frac{\partial H_\gamma}{\partial x_1} + \frac{\partial H_\gamma}{\partial x_2}\right) = \delta(0)$$

along the half line $(0,0,t)$, $t>0$.

Since H_γ is a constant multiple of H, that H satisfies all of the conditions of $\tilde{P}_\gamma$ except the corner condition clearly implies that H_γ does also. Hence we have proved.

THEOREM 3.4 : H_γ *is a solution to* $\tilde{P}_\gamma$ *with*

$$g = \delta(t).$$

We define

$$v_\gamma = H_\gamma * g_\gamma, \tag{3.44}$$

where g_γ is defined by (3.35), and define

$$u_\gamma = v_\gamma + u^*. \tag{3.45}$$

Then, using the properties of convolution with respect to derivatives, we have the following corollaries :

COROLLARY 3.3 : v_γ *is a solution to* $\tilde{P}_\gamma$ *with*

$$g = g_\gamma.$$

COROLLRAY 3.4 : u_γ *is a solution to* P_γ.

It is interesting to determine a posteriori the behavior of u_γ at the corner $x_1=x_2=0$.

LEMMA 3.5 : $u_\gamma(0,0,t)$ *is proportional to* $u^*(0,0,t)$*; thus* u_γ *is* C^∞ *in time along* $(0,0,t)$.

Proof : Clearly it suffices to show that $v_\gamma = u_\gamma - u^*$ is proportional to u^* along $x_1=x_2=0$. By definition

$$v_\gamma(0,0,t) = (\gamma-\gamma^*) \int_0^t H(0,0,t-s) \frac{\partial^2 u^*}{\partial t^2}(0,0,s)\, ds \;;$$

however, from lemma 3.4, $H_\gamma(0,0,t-s) = a_\gamma\, c_1(t-s)$. Thus

$$v_\gamma(0,0,t) = a_\gamma\, c_1(\gamma-\gamma^*) \int_0^t (t-s) \frac{\partial^2 u^*}{\partial t^2}(0,0,s)\, ds$$

$$= a_\gamma\, c_1(\gamma-\gamma^*)\, u^*(0,0,t).$$

<u>*THEOREM 3.5*</u> *The second order spatial derivatives of* u_γ *are singular at the corner* $x_1=x_2=0$ *for* $t>0$ *for* $\gamma\neq\gamma^*$.

<u>*Proof*</u> *:* It sufficies to show that v_γ does not satisfy the wave equation along the axis $(0,0,t)$. We have :

$$\frac{\partial^2}{\partial x_j^2} v_\gamma(0,0,t) = a_\gamma(\gamma-\gamma^*) \int_0^t \frac{\partial^2 H}{\partial x_j^2}(0,0,t-s) \frac{\partial^2 u^*}{\partial t^2}(0,0,s)\, ds, \qquad j=1,2.$$

Moreover, as H satisfies the wave equation in the classical sense on $(0,0,t)$, $t>0$ and is symmetric in x_1 and x_2, we have :

$$\frac{\partial^2 H}{\partial x_j^2}(0,0,t) = \frac{1}{2}\frac{\partial^2 H}{\partial t^2}(0,0,t) = \frac{d^2}{dt^2}\left(\frac{1}{2}\; c_1 t\right) = 0.$$

Hence $\frac{\partial^2}{\partial x_j^2} v_\gamma(0,0,t-s) = 0$, $j=1,2$. However, as a consequence of Lemma 3.5.

$$\frac{\partial^2}{\partial t^2} v_\gamma(0,0,t) = a_\gamma\, c_1(\gamma-\gamma^*) \frac{\partial u^*}{\partial t^2}(0,0,t).$$

Thus,

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) v_\gamma(0,0,t) = a_\gamma\, c_1(\gamma-\gamma^*) \frac{\partial^2}{\partial t^2} u^*(0,0,t), \qquad t>0.$$

As $\frac{\partial^2 u^*}{\partial t^2}$ is not identically zero along the half line $(0,0,t)$, $\left(\frac{\partial^2}{\partial t} - \Delta\right) v_\gamma(0,0,t)$ is identically 0 only when $\gamma=\gamma^*$. □

Concerning the behavior of u_γ along the boundary of the cone we have :

<u>*THEOREM 3.6*</u> *The second order time derivatives of* u_γ *are continuous but the second order spatial derivatives are singular along* $x_1^2+x_2^2=t^2$, $t>0$.

Proof : It clearly suffices to consider v_γ as u^* is C^∞. Then we have only to note that the convolution in time defining v_γ makes u_γ C^2 in time along $x_1^2+x_2^2=t^2$ as H_γ is C^1 in time there. However, as H_γ is only C^1 in space along $x_1^2+x_2^2=t^2$, v_γ is also.

Remarks :

- The function H_γ can be shown to be of finite energy (Indeed for each $t>0$, H_γ is in $C^1(\bar{\Omega})$ and compactly supported). As it is initially 0 in Ω, H_γ gives a counter example to the uniqueness of the solution to (P) (i.e. without the corner condition) in the class of finite energy solutions.
- It can also be shown that the solution u_γ defined by (3.45) belongs to the class of functions for which we get a uniqueness result in section 3.2.

In summary we have shown concerning the corner wave v_γ :

- v_γ is a solution of the wave equation in Ω in the distributionnal sense.
- v_γ satisfies the corner condition $CC\gamma$.
- v_γ has singular second order spatial derivatives along $|x| = t$ and along $|x| = 0$.
- v_γ is of finite energy.

Of course, the boundary conditions CL1 and CL2, as well as the corner condition ($CC\gamma$) are satisfied in the weak sense of Definition 3.1.

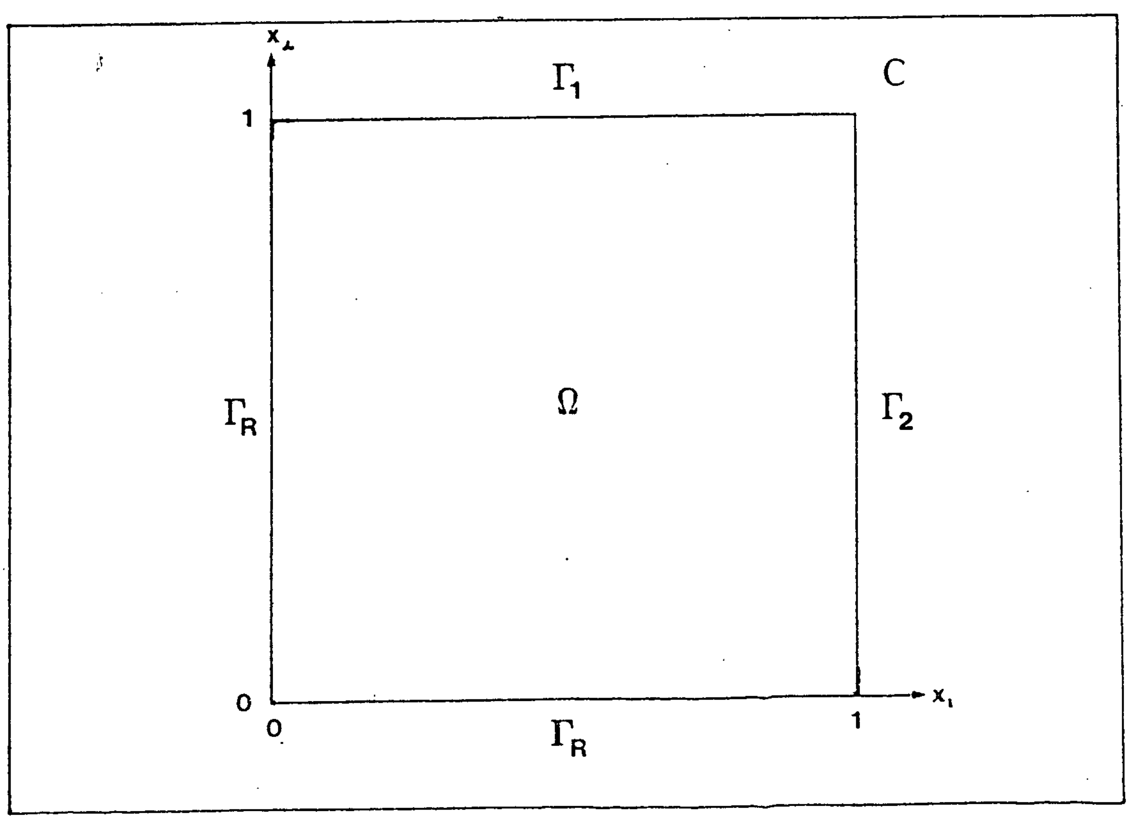
IV - NUMERICAL RESULTS

Here we exhibit the results of numerical experiments confirming the conclusions obtained in the previous section. These results were obtained using a variational scheme which we apply not to the wave equation (1.1) itself but to its time derivative in order to be able to use the boundary equations after the standard integration by parts in the variational formulation. The discretization in time is obtained using an explicit scheme incorporating the four time levels n-2, n-1, n, and n+1 as a third order time derivative appears in the time differentiated wave equation. We remark that it is not the numerical scheme per se that interests us here and that another scheme, the finite difference scheme proposed by Engquist and Majda in [5] for example, could just as well have been used.

IV .1 - Numerical scheme

IV.1.1 - Variational formulation

We let Ω denote the unit square $(0,1)\times(0,1)$ with boundary Γ written as the disjoint union of Γ_1 the upper edge, Γ_2 the right hand edge, C the upper right hand corner, and Γ_R the left and lower edges with the three remaining corners.



The problem we shall solve numerically is the following :

$$\frac{\partial^3 u}{\partial t^3} - \frac{\partial \Delta u}{\partial t} = \frac{\partial f}{\partial t}, \qquad \text{in } \Omega, \tag{4.1}$$

with boundary condition

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t \partial x_2} - \frac{1}{2}\frac{\partial^2 u}{\partial x_1^2} = 0, \qquad \text{in } \Gamma_1, \tag{4.2$_1$}$$

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t \partial x_1} - \frac{1}{2}\frac{\partial^2 u}{\partial x_2^2} = 0, \qquad \text{in } \Gamma_2, \tag{4.2$_2$}$$

$$\gamma \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = \lambda(t), \qquad \text{at } C, \tag{4.3}$$

$$u = 0, \qquad \text{on } \Gamma_R, \tag{4.4}$$

and initial conditions

$$\begin{cases} u = u_0, & \text{at } t = 0, \\ \dfrac{\partial u}{\partial t} = u_1, & \text{at } t = 0, \end{cases} \tag{4.5}$$

where $f \in L^2(\Omega)$ is of compact support.

The solution u will be sought in the space V of functions in $H^1(\Omega)$ having trace in $H_0^1(\Gamma)$, where by $H_0^1(\Gamma)$ we shall denote the Hilbert space of functions in $L^2(\Gamma)$ whose restrictions to Γ_1 and to Γ_2 are H^1 functions determining the same value at the corner C and whose restriction to Γ_R is identically 0, (see III.1) :

$$V = \left\{ v \in H^1(\Omega) \; : \; \text{trace } (u) \in H_0^1(\Gamma) \right\}$$

$$H_0^1(\Gamma) = \left\{ \phi \in L^2(\Gamma) \; : \; \phi_1 = \phi_{|\Gamma_1} \in H^1(\Gamma_1) \; ; \; \phi_2 = \phi_{|\Gamma_2} \in H^1(\Gamma_2) \; ; \; \phi_1(C) = \phi_2(C) \; ; \text{ and } \phi_{|\Gamma_R} = 0 \right\}.$$

Proceeding as in section III.1 we can write the variational formulation of problem (4.1), $(4.2)_1$, $(4.2)_2$, (4.3), (4.4), (4.5) :

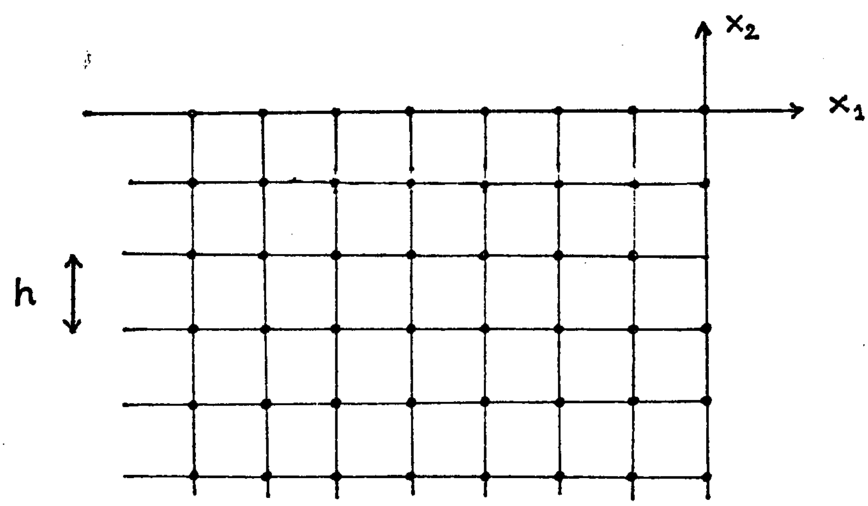
Find $u \in V$ such that :

$$(4.6)\quad \left| \begin{array}{l} \frac{\partial^3}{\partial t^3}(u,v) + \frac{\partial}{\partial t}(\nabla u, \nabla v) + \frac{\partial^2}{\partial t^2}\langle u,v\rangle_\Gamma + \frac{1}{2}\langle \frac{\partial u}{\partial x_1}, \frac{\partial v}{\partial x_1}\rangle_{\Gamma_1} + \frac{1}{2}\langle \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_2}\rangle_{\Gamma_2} \\ + \frac{1}{2}\gamma \frac{\partial}{\partial t} u(1,1)\, v(1,1) = \frac{\partial}{\partial t}(f,v) + \frac{1}{2}\lambda\, v(1,1) \quad , \quad v \in V. \end{array} \right.$$

IV.I.2 - Spatial discretization

To discretize Ω we use a uniform mesh of triangles obtained by cutting along the upper left to lower right diagonal each of the squares of a uniform mesh of squares. Let n be a positive integer, $h = \frac{1}{n}$, and $x_{ij} = (ih, jh)$, $i = 0, \dots, n$; $j = 0, \dots, n$. Then our mesh T_h contains all of the triangles $(x_{ij}, x_{i+1,j}, x_{i+1,j-1})$ and $(x_{ij}, x_{i,j-1}, x_{i+1,j-1})$ $i=0,\dots, n-1$; $j=1, \dots, n$.

The approximation u_h to u will be sought in the space V_h of functions in V whose restriction to each triangle of T_h is a linear function. Thus V_h shall have a basis of functions $\left\{v_{ij} : i=1,\dots,n,\ j=1,\dots,n\right\}$ where v_{ij} has value 1 at x_{ij} and vanishes at all other nodes of the grid, and u_h may be expressed as $\sum_{i=1}^{n} u_{ij}\, v_{ij}$ where $u_{ij} = u_h(x_{ij})$.



We can now write the problem discretized in space as follows :

Find $u_h \in V_h$ such that :

$$\frac{\partial^3}{\partial t^3}(u_h, v_{ij}) + \frac{\partial}{\partial t}(\nabla u_h, \nabla v_{ij}) + \frac{\partial^2}{\partial t^2}<u_h, v_{ij}>_\Gamma + \frac{1}{2}<\frac{\partial u_h}{\partial x_1}, \frac{\partial v_{ij}}{\partial x_1}>_{\Gamma_1}$$

$$+ \frac{1}{2}<\frac{\partial u_h}{\partial x_2}, \frac{\partial v_{ij}}{\partial x_2}>_{\Gamma_2}, + \frac{1}{2}\gamma \frac{\partial}{\partial t} u_h(1,1)\, v_{ij}(1,1)$$

$$= \frac{\partial}{\partial t}(f, v_{ij}) + \frac{1}{2}\lambda\, v_{ij}(1,1), \qquad i,j = 1,\dots,n.$$

The integrals occuring in the first and third terms of the left hand side as well as the integral in the first term of the right hand side will be computed numerically using mass lumping whereas all other integrals will be computed exactly. The resulting equations are :

$$h^2 \frac{\partial^3}{\partial t^3} u_{ij} + \frac{\partial}{\partial t}\left\{4u_{ij} - u_{i-1,j} - u_{i,j-1} - u_{i+1,j} - u_{i,j+1}\right\} = h^2 \frac{\partial}{\partial t} f(x_{ij}) \qquad 1 \le i,j < n,$$

$$\frac{h^2}{2}\frac{\partial^3}{\partial t^3} u_{in} + \frac{\partial}{\partial t}\left\{2u_{in} - u_{i,n-1} - \frac{1}{2}u_{i-1,n} - \frac{1}{2}u_{i+1,n}\right\} +$$

$$+ h\frac{\partial^2}{\partial t^2} u_{in} + \frac{1}{2h}\left\{2u_{in} - u_{i-1,n} - u_{i+1,n}\right\} = 0 \qquad 1 \le i < n,$$

$$\frac{h^2}{2}\frac{\partial^3}{\partial t^3} u_{nj} + \frac{\partial}{\partial t}\left\{2u_{nj} - u_{n-1,j} - \frac{1}{2}u_{n,j-1} - \frac{1}{2}u_{n,j+1}\right\}$$

$$+ h\frac{\partial^2}{\partial t^2} u_{nj} + \frac{1}{2h}\left\{2u_{nj} - u_{n,j-1} - u_{n,j+1}\right\} = 0 \qquad 1 \le j < n,$$

$$\frac{h^2}{6}\frac{\partial^3}{\partial t^3} u_{nn} + \frac{\partial}{\partial t}\left\{u_{nn} - \frac{1}{2}u_{n-1,n} - \frac{1}{2}u_{n,n-1}\right\}$$

$$+ h\frac{\partial^2}{\partial t^2} u_{nn} + \frac{1}{2h}\left\{2u_{nn} - u_{n-1,n} - u_{n,n-1}\right\} + \frac{\gamma}{2}u_{nn} = \frac{1}{2}\lambda.$$

IV.I.3 - Time discretization

The time discretization is an explicit four level scheme involving times m,m+1,m-1, and m-2 and is centered at time m-1/2. The third derivative is approximated by taking the difference over Δt between the standard centered second order difference at level m and that at level m-1:

$$\frac{\partial^3}{\partial t^3}(\cdot) \sim \frac{1}{\Delta t^3}\left\{(\cdot)^{m+1} - 3(\cdot)^m + 3(\cdot)^{m-1} - (\cdot)^{m-2}\right\}.$$

The second derivative is approximated by the average of the standard centered second order differences at levels m and m-1 :

$$\frac{\partial^2}{\partial t^2}(\cdot) \sim \frac{1}{2\Delta t^2}\left\{(\cdot)^{m+1} - (\cdot)^m - (\cdot)^{m-1} + (\cdot)^{m-2}\right\}.$$

The first order derivative is approximated by a simple centered difference (at level $m-\frac{1}{2}$) :

$$\frac{\partial}{\partial t}(\cdot) \sim \frac{1}{\Delta t}\left\{(\cdot)^m - (\cdot)^{m-1}\right\},$$

and finally terms not involving a time derivative are averaged between time levels m and m-1 :

$$(\cdot) \sim \frac{1}{2}\left\{(\cdot)^m + (\cdot)^{m-1}\right\}.$$

We initialize by putting

$$u^1_{ij} = u_0(x_{ij}), \qquad i,j = 1, \ldots, n,$$

$$u^2_{ij} = \Delta t\, u_1(x_{ij}) + u_0(x_{ij}), \qquad i,j = 1, \ldots, n.$$

Then for $m \geq 2$, letting $g = \Delta t/h$, we solve

- in the interior

$$u_{ij}^{m+1} = 3u_{ij}^{m} - 3u_{ij}^{m-1} + u_{ij}^{m-2} +$$

$$+ g^2 \Big\{ \big[u_{i-1,j}^{m} + u_{i,j-1}^{m} + u_{i+1,j}^{m} + u_{i,j+1}^{m} - 4u_{ij}^{m}\big]$$

$$- \big[u_{i-1,j}^{m-1} + u_{i,j-1}^{m-1} + u_{i+1,j}^{m-1} + u_{i,j+1}^{m-1} - 4u_{ij}^{m-1}\big]\Big\}$$

$$+ \Delta t^2 \,(f_{ij}^{m} - f_{ij}^{m-1}) \quad , \qquad 1 \le i,j < n,$$

- on the upper boundary

$$(1+g)\, u_{in}^{m+1} = (3+g)\, u_{in}^{m} - (3-g)\, u_{in}^{m-1} + (1-g)\, u_{in}^{m-2}$$

$$+ g^2 \Big\{\big[\, 2u_{i,n-1}^{m} + (1+g/2)\, u_{i-1,n}^{m} + (1+g/2)\, u_{i+1,n}^{m} - (4+g)\, u_{in}^{m}\big]$$

$$- \big[\, 2u_{i,n-1}^{m-1} + (1+/2)\, u_{i-1,n}^{m-1} + (1+g/2)\, u_{i+1,n}^{m-1} - (4+g)\, u_{in}^{m-1}\big]\Big\},$$

$$i=1,\dots,n,$$

- on the right hand boundary

$$(1+g)\, u_{nj}^{m+1} = (3+g)\, u_{nj}^{m} - (3-g)\, u_{nj}^{m-i} + (1-g)\, u_{nj}^{m-2}$$

$$+ g^2 \Big\{\big[\, 2u_{n-1,j}^{m} + (1+g/2)\, u_{n,j-1}^{m} + (1+g/2)\, u_{n,j+1}^{m} - (4+g)\, u_{nj}^{m}\big]$$

$$- \big[\, 2u_{n-1,h}^{m-1} + (1+g/2)\, u_{n,j-1}^{m-1} + (1+g/2)\, u_{n,j+1}^{m-1} - (4+g)\, u_{nj}^{m-1}\big]\Big\},$$

$$j=1,\dots,n,$$

- at the corner

$$(2/3+2g)\, u_{nn}^{m+1} = (2+2g)\, u_{nn}^{m} - (2-2g)\, u_{nn}^{m-1} + (2/3-2g)\, u_{nn}^{m-2}$$

$$+ g^2 \Big\{\big[\,(2+g)\, u_{n,n-1}^{m} + (2+g)\, u_{n-1,n}^{m} - (4+2g)\, u_{nn}^{m}\big]$$

$$- \big[\,(2+g)\, u_{n,n-1}^{m-1} + (2+g)\, u_{n-1,n}^{m-1} - (4+2g)\, u_{nn}^{m-1}\big]\Big\}$$

$$+ g^2\, \Delta t\, (\lambda^{m} + \lambda^{m-1}).$$

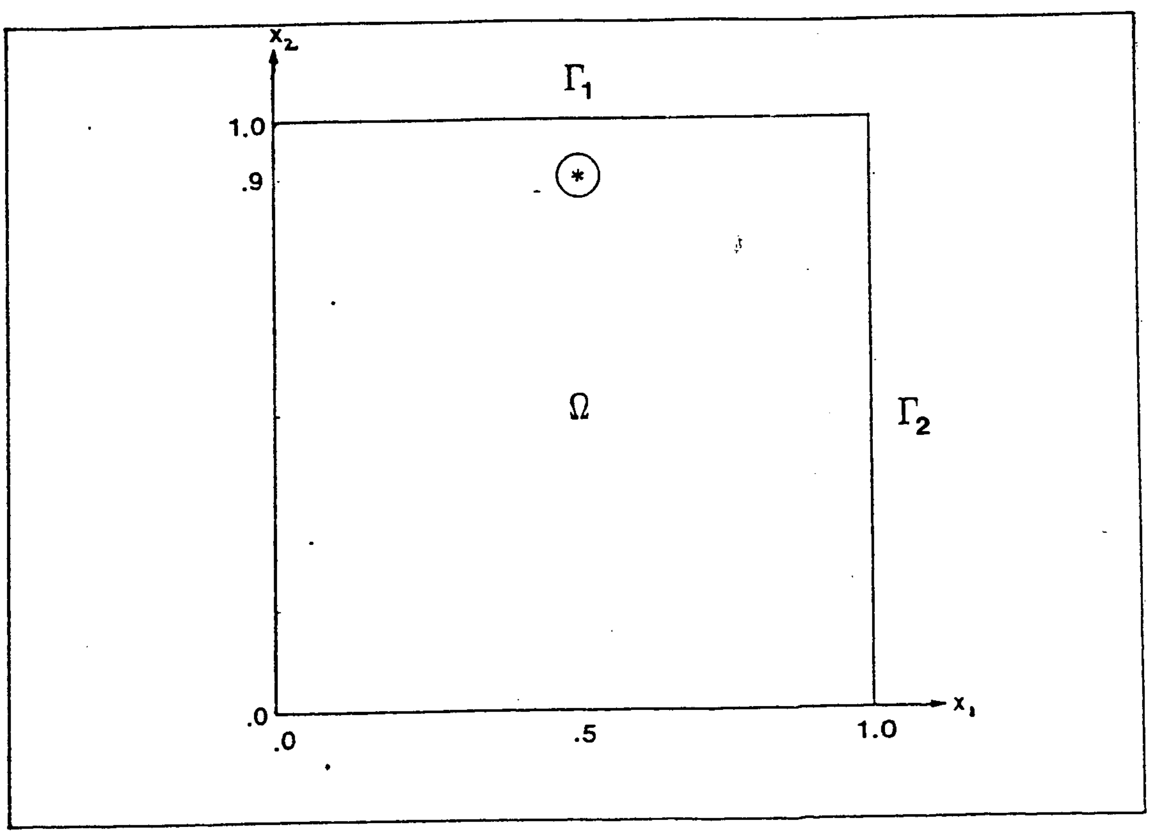
V.2 - Description of experiments

V.2.1 - Study of a single absorbing boundary

In the first set of experiments, we observe the effect of introducing an artificial boundary with the second order absorbing boundary condition.

The results obtained with the 2nd order condition are compared with those obtained when no artificial boundary is introduced as well as with those produced when a 1st order condition is used for the absorbing boundary. It is also interesting to include the results given by the Neumann boundary condition, a totally reflecting condition.

Our domain of observation is the unit square and the calculations have been done with mesh size $\Delta x = .01$ and time step $\Delta t = .005$. The source was placed near the upper edge with center at (.5, .9) and radius .04 so that the other edges of the square are not attained by the wave at the times of observation t=.3 and t=.5.



For the 2nd order condition we have taken the domain and boundary conditions exactly as described in secion IV.1. For the 1st order condition (4.2) is replaced by

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial t \partial x_2} = 0 & \text{on } \Gamma_1, \\[2ex] \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial^2 u}{\partial t \partial x_1} = 0 & \text{on } \Gamma_2, \end{cases}$$

and (4.6) is replaced by

$$\frac{\partial^3}{\partial t^3}(u,v) + \frac{\partial}{\partial t}(\nabla u, \nabla v) + \frac{\partial^2}{\partial t^2}\langle u,v\rangle_\Gamma = \frac{\partial}{\partial t}(f,v), \qquad v \in V,$$

whereas for the Neumann condition (4.7) becomes

$$\begin{cases} \dfrac{\partial u}{\partial x_2} = 0 & \text{on } \Gamma_1, \\[2ex] \dfrac{\partial u}{\partial x_1} = 0 & \text{on } \Gamma_2, \end{cases}$$

so that instead of (4.6) we have :

$$\frac{\partial^3}{\partial t^3}(u,v) + \frac{\partial}{\partial t}(\nabla u, \nabla v) = \frac{\partial}{\partial t}(f,v).$$

The so called "exact" solution, is obtained by doing the calculations on the rectangle $[0,2]\times[0,2]$ so that the wave has not reached any edge at the observation times and there is thus no reflection.

The source has been introduced as a right hand side

$$f(x_1,x_2,t) = g(x_1,x_2)\, h(t).$$

$$g(x_1,x_2) = \begin{cases} 10\,000 \times (1-r/.04) & \text{if } r < .04, \\[2ex] 0 & \text{otherwise}, \end{cases}$$

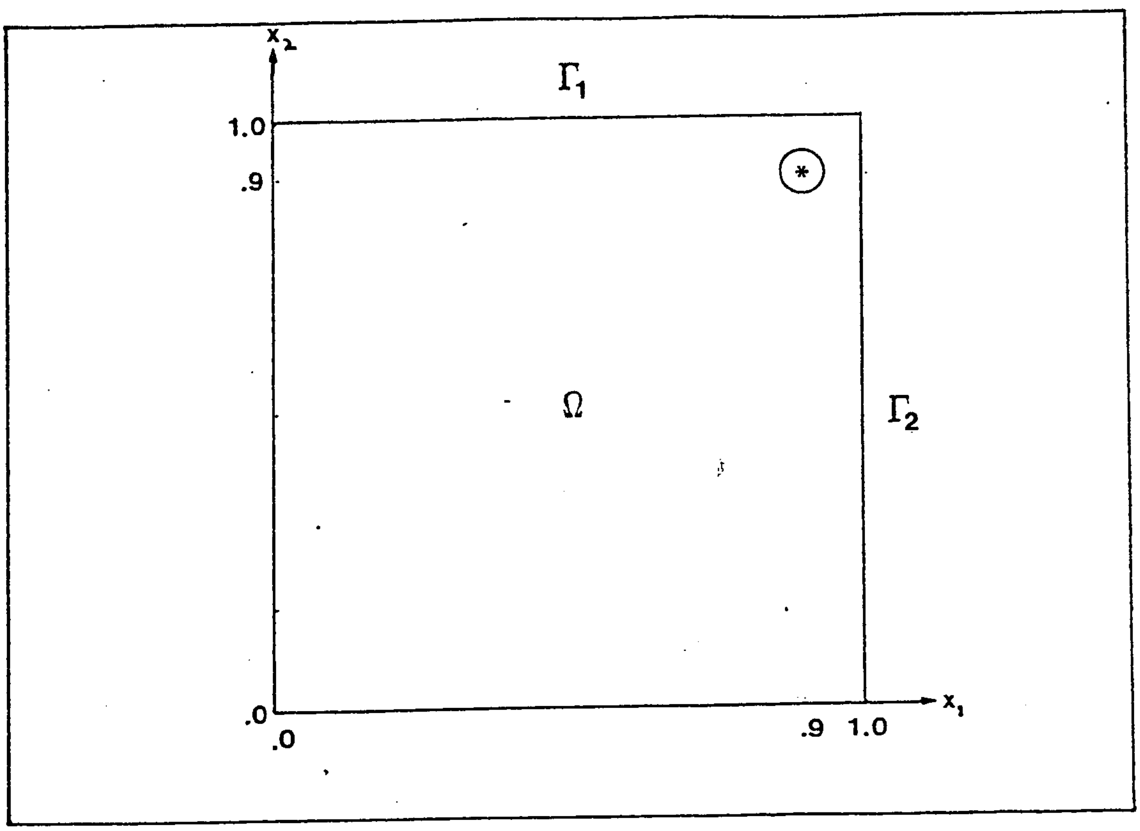
where r is the distance of (x_1,x_2) from the center of the source (.5, .9)

$$h(t) = \begin{cases} e^{-10(1-t/.05)^2} & \text{if } t < .1 \\[2ex] 0 & \text{otherwise}. \end{cases}$$

V.2.2 - Study of the corner

The remaining experiments, are concerned with the reflection by the corner. We study the corner condition (4.3) for several choices of the constant γ, γ=1.5, γ=0.1, and γ=3.0. The results obtained are compared among themselves as well as with those obtained in the ideal case when no artificial boundary is introduced. Here also it is interesting to observe the results obtained when a first order condition or the Neumann condition is used for the edges, boundary conditions for which no special condition need be specified at the corner.

Again our domain of observation is the unit square but here the source, while of exactly the same form as in the previous experiments, is located near the upper right corner with center at (.9, .9).



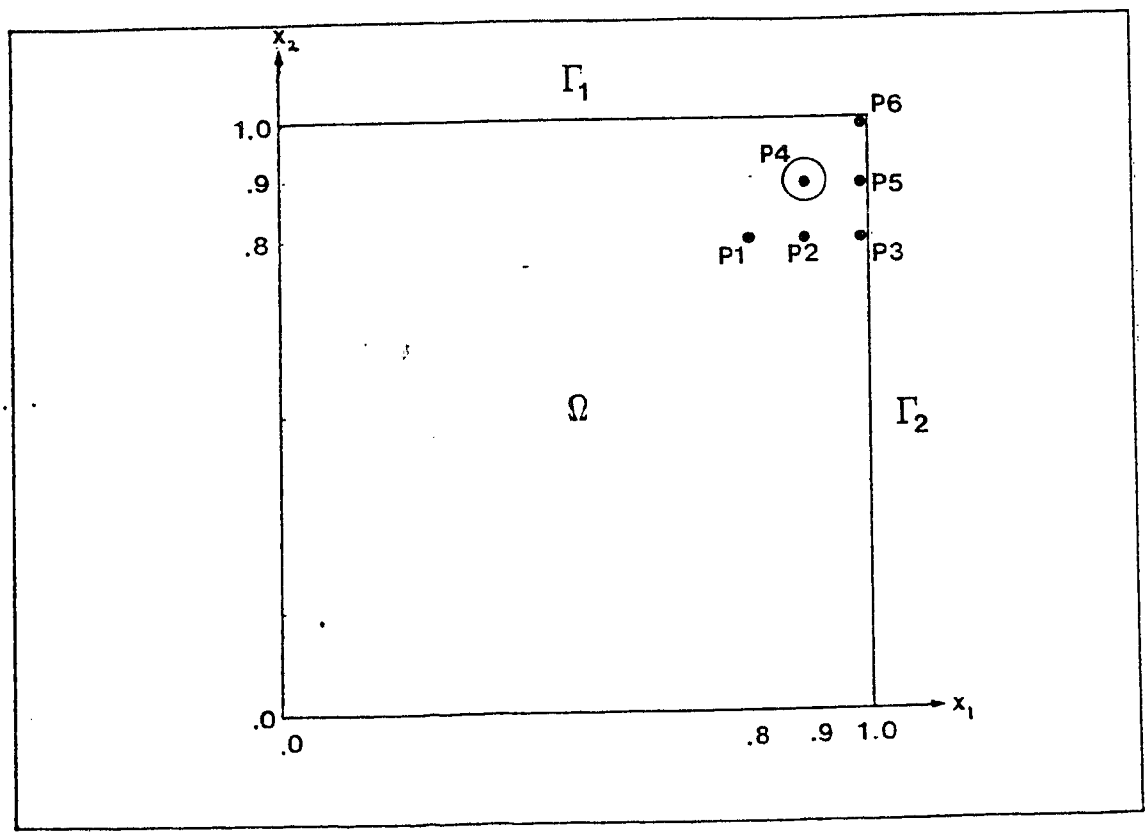
As before for the second order condition, the domain and boundary conditions are as given in section IV.1 while the same modifications as indicated above are used in the cases of no boundary, of a first order boundary, and of a Neumann boundary.

For most of the results we have used the same discretization as earlier $\Delta x=.01$, $\Delta t=.005$ (discretization 1) but in order to see that the γ-wave is not a discretization phenomena we have also presented results obtained with a finer mesh, $\Delta x = .005$, $\Delta t=.0025$ (discretization 2).

For each of the experiments we have presented "snapshots" representing the solution u as a function of (x_1,x_2) at the times $t=.3$ and $t=.5$. For the experiments concerning the corner, where condition (4.3) for the various choices of γ is used we have also given seismograms representing the solution u as a function of t at several points near the corner. The seismograms were actually obtained by repeating these experiments and the source was placed at (.89, .89) instead of (.9, .9). The points where the response was calculated are

$$P_1 = (.79, .79) \quad , \quad P_2 = (.89, .79) \quad , \quad P_3 = (.99, .79)$$

$$P_4 = (.89, .89) \text{ the source, } P_5 = (.99, .89), \; P_6 = (.99, .99).$$



IV.3 - Results of experiments

IV.3.1 - Experiments concerning a single absorbing boundary

In figures 1 through 14 we see the results of the experiments concerned with a single absorbing side. In these experiments we are essentially concerned with comparing the behavior of the first and second order absorbing conditions to see how much each of these differs from an ideal condition which would act as if there were no boundary. However, for completeness we have also included the results obtained with a totally reflecting condition, the Neumann condition.

Figures 1 and 2 give the so called exact solution, i.e the solution obtained on a larger domain for which the side of the square under consideration was actually interior, at times $t=.3$ and $t=.5$. In figures 3 and 4 we present the solution obtained with the second order absorbing boundary condition at these same times, in figures 5 and 6 those obtained with the first order absorbing boundary condition and in figures 7 and 8 those obtained with the Neumann condition.

First observe that for all of these figures, except 7 and 8 where the Neumann condition was used, the minimum and maximum are the same for the figures representing the solutions at the corresponding times, so that the scale is the same in figures 1, 3, and 5 and again in figures 2, 4, and 6. Also figures 1, 3, and 5, or again 2, 4, and 6, at first glance seem to have roughly the same form. However, on closer examination, in figures 5 and 6 one can see a trace of the reflected wave seen in figures 7 and 8 though here in the opposite direction.

To better see the waves reflected by the different boundary conditions we present in figures 9 through 14 the differences between the solutions obtained with the various boundary conditions and the exact solution. We emphasize that the scales for these figures are not at all the same. In figures 9 and 10 representing the wave reflected by the second order condition the amplitude is roughly half that in figures 11 and 12 representing the wave reflected by the first order condition. We remark that in both cases the amplitude of the reflection increases with the angle θ from normal incidence; however in figures 9 and 10 the amplitude is "small" for a much larger range of values of θ. Thus a noticeable disturbance propagates into the interior of the domain much more pronouncedly and more rapidly with the first order condition than with the second order condition. This is, of course, as expected since the reflection coefficients R_2 and R_1 for the second and first order conditions are given by :

$$R_2 = -\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^2$$

$$R_1 = -\left(\frac{1-\cos\theta}{1+\cos\theta}\right) .$$

For the Neumann condition the reflection coefficient R_N is 1 indicating the perfect reflection of the initial wave seen in figures 13 and 14.

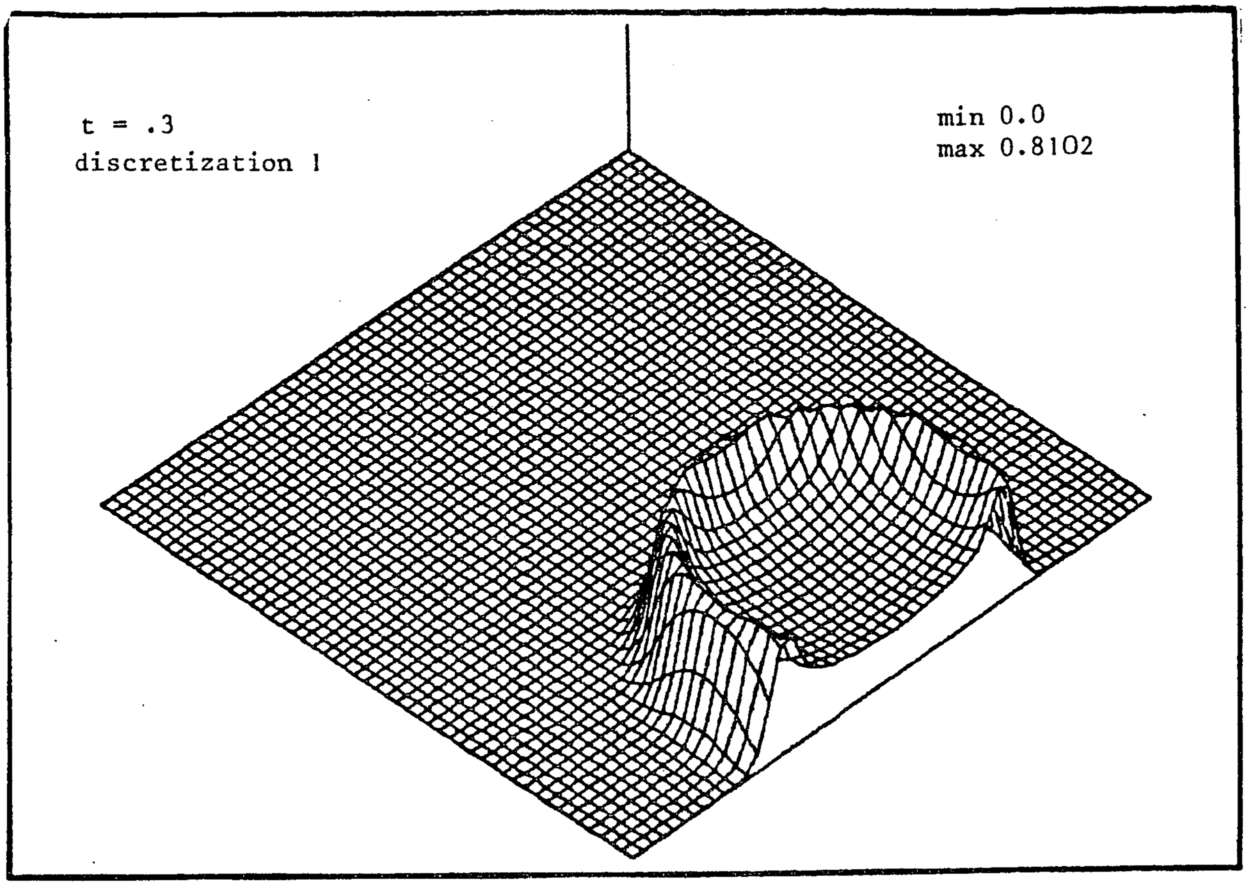


Figure 1 : exact solution

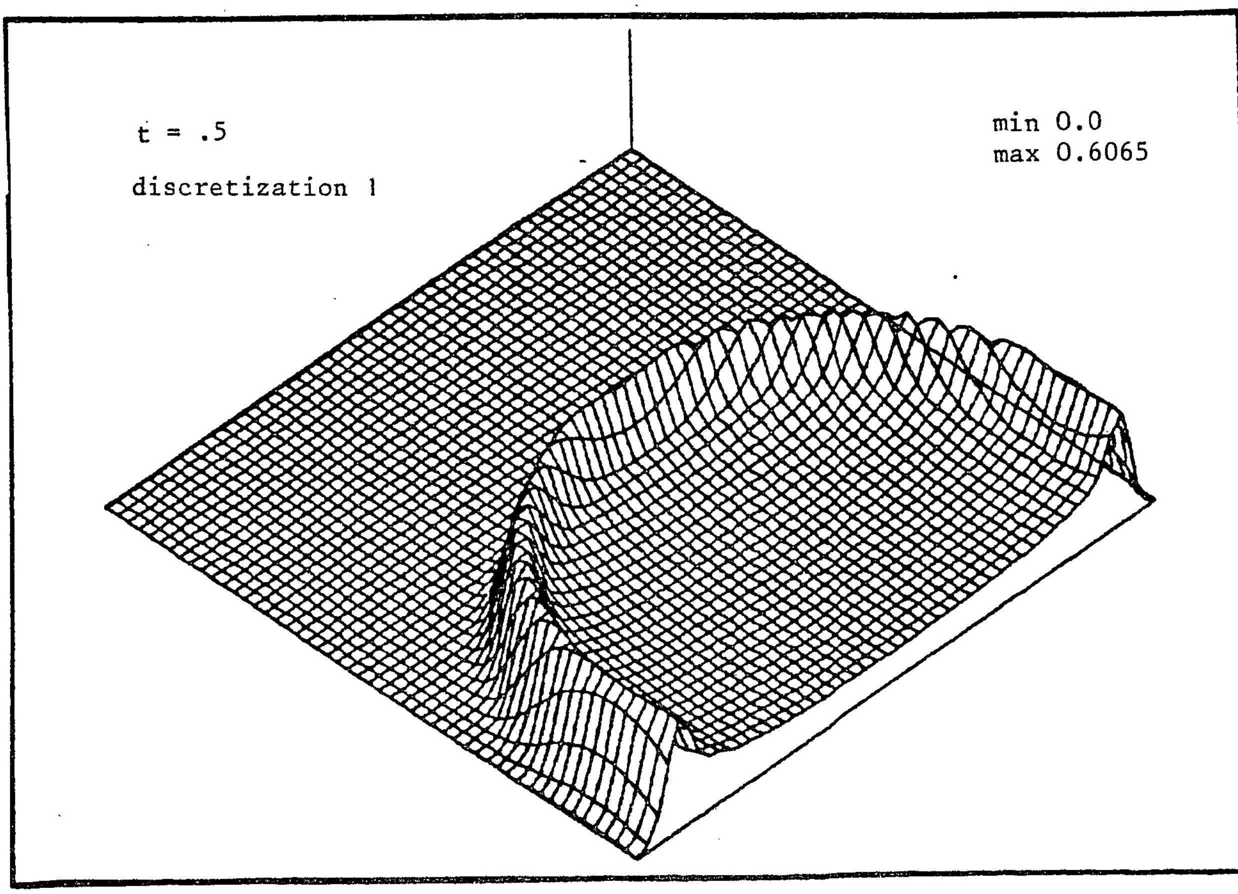


Figure 2 : exact solution

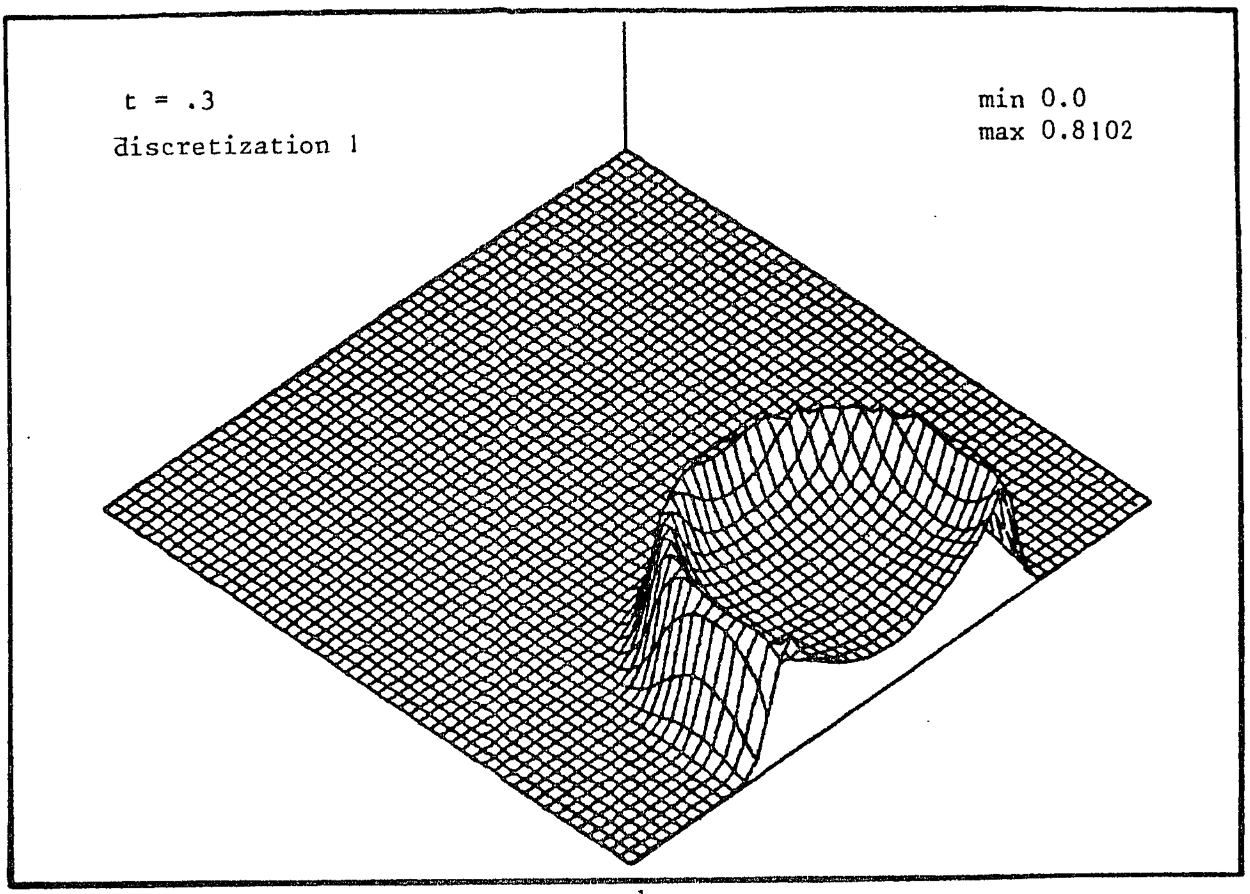


Figure 3 : solution calculated with 2^{nd} order boundary condition

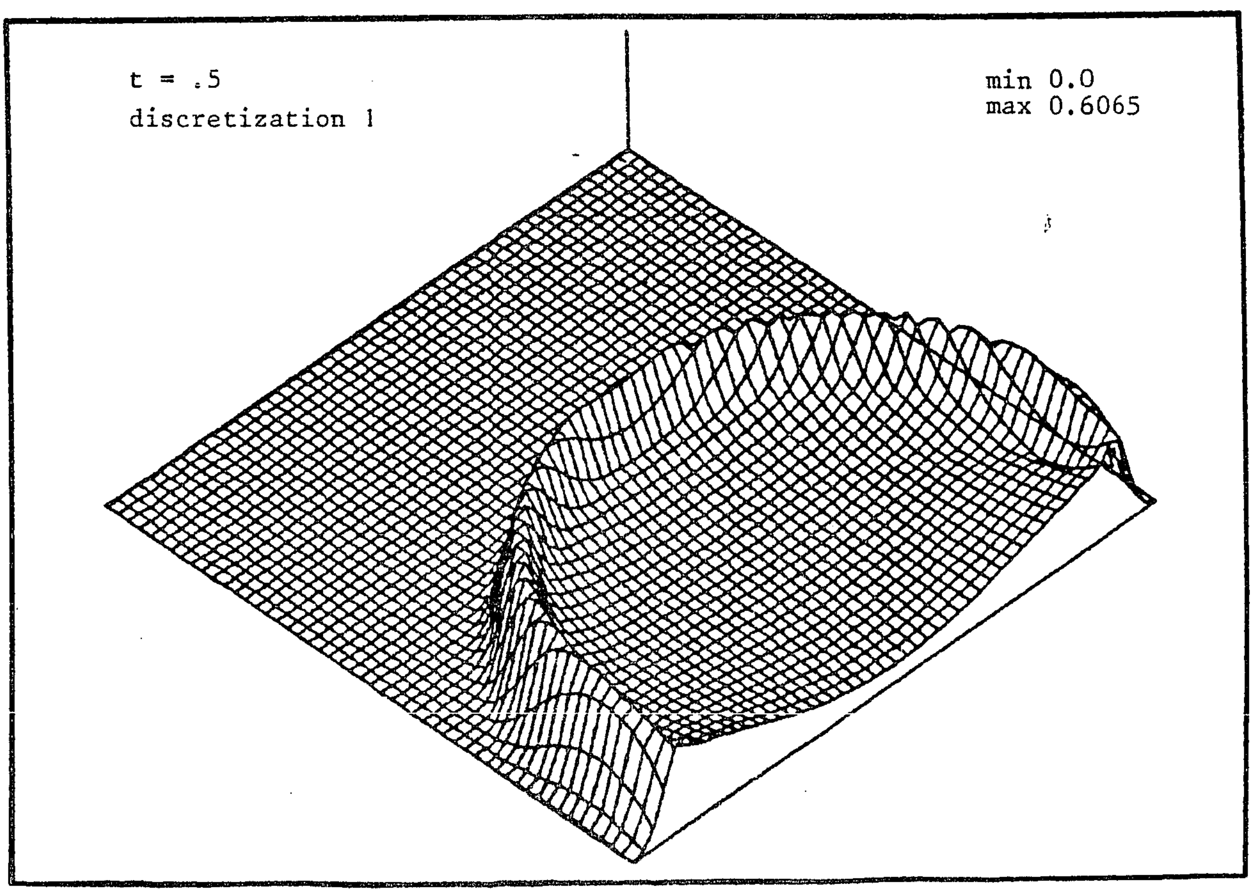


Figure 4 : solution calculated with 2^{nd} order boundary condition

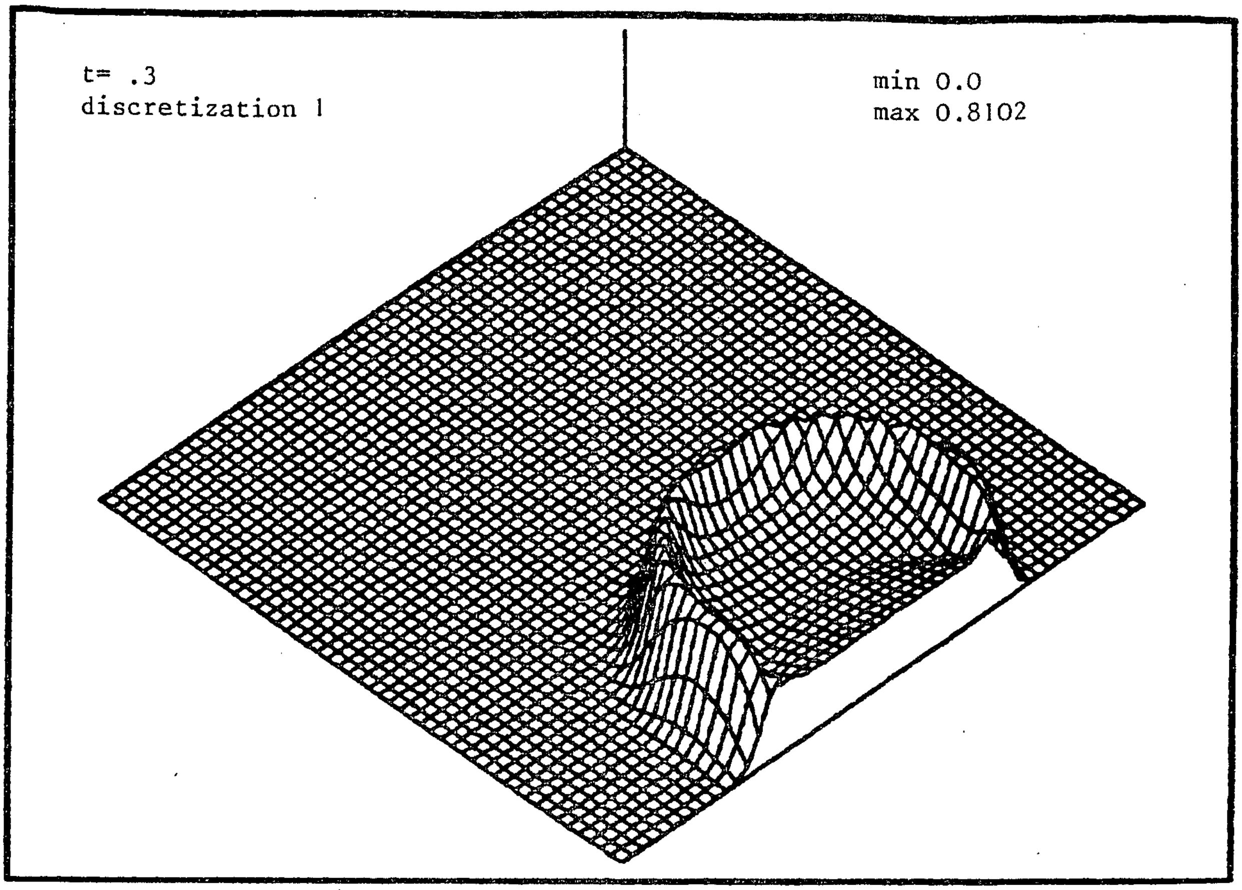


Figure 5: Solution calculated with 1st order boundary condition

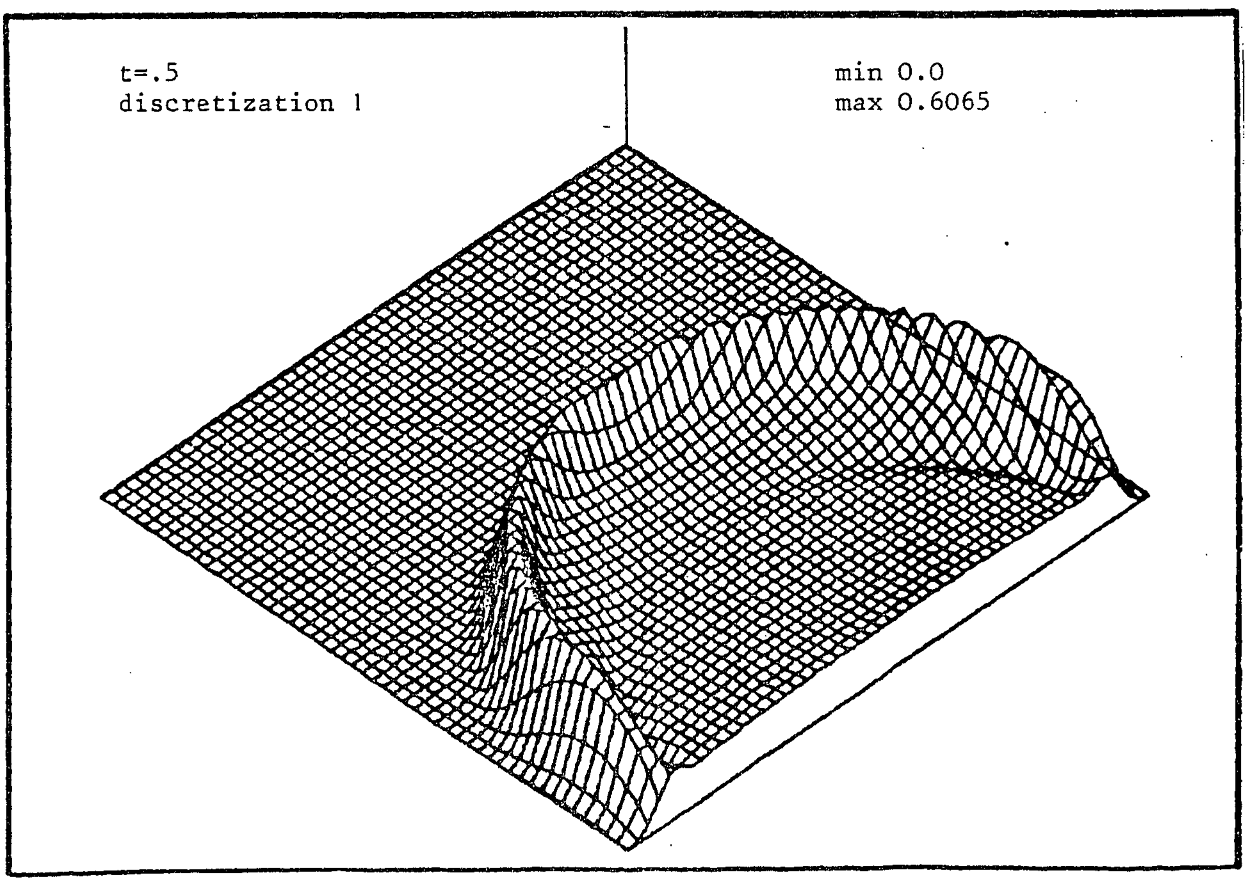


Figure 6: Solution calculated with 1st order boundary condition

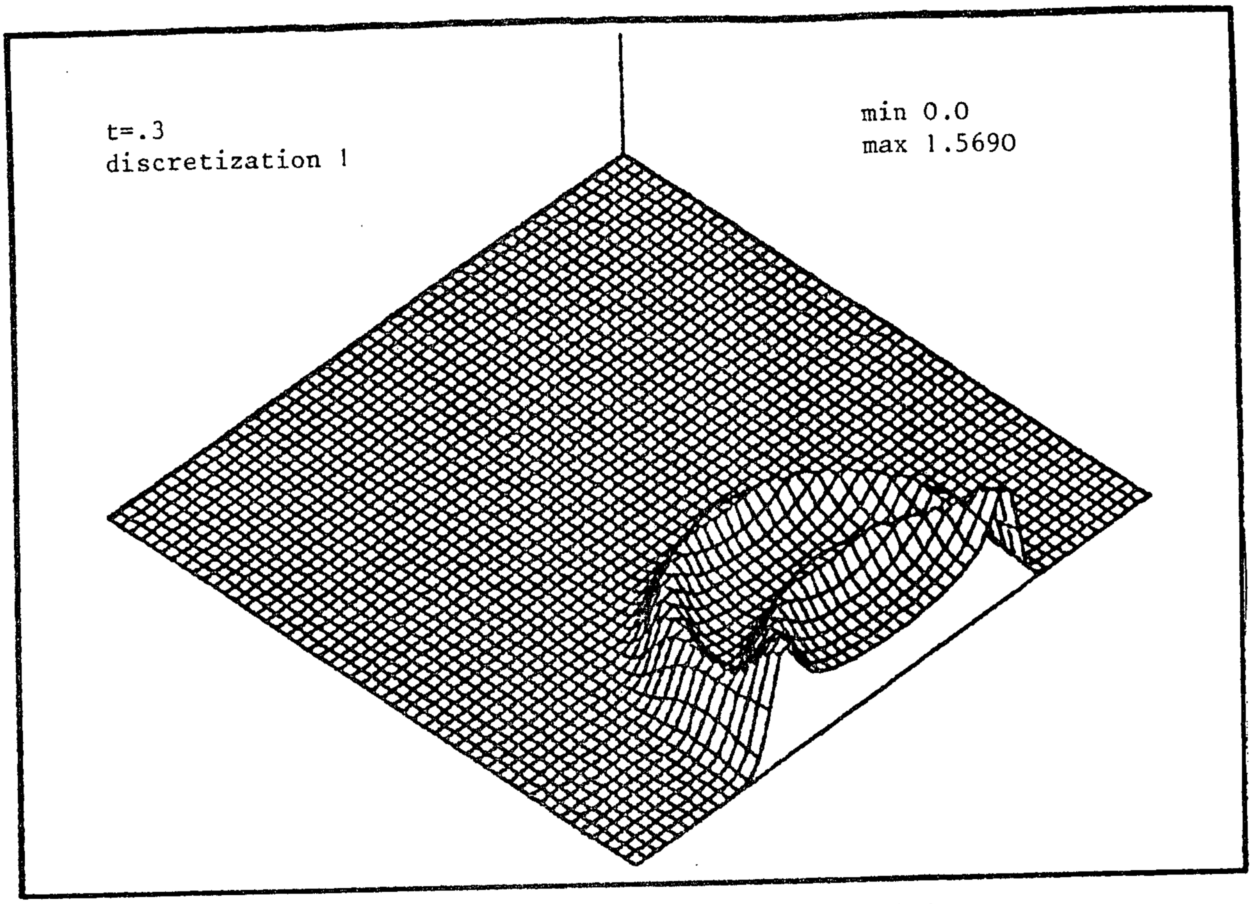


Figure 7: Solution calculated with Neumann boundary condition.

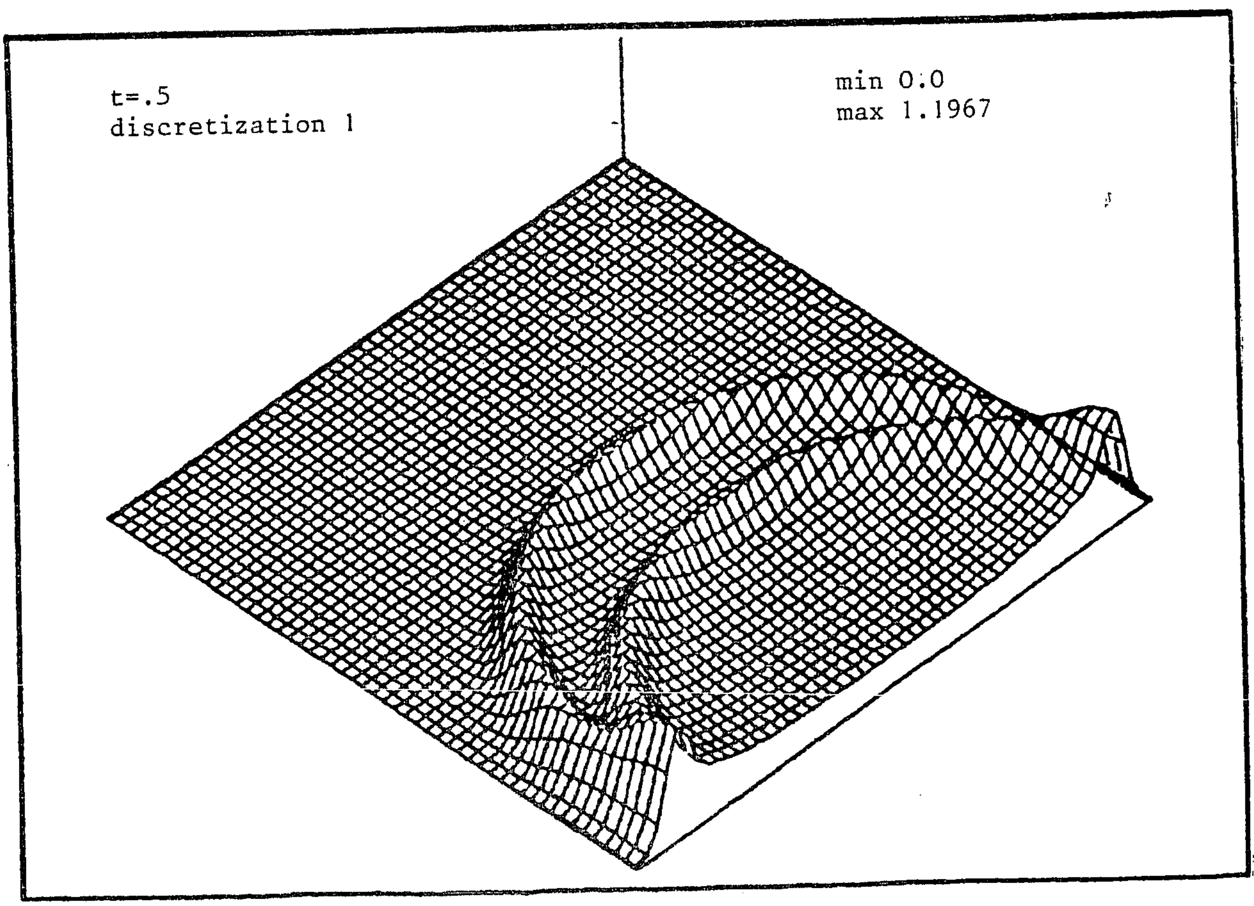


Figure 8: Solution calculated with Neumann boundary condition.

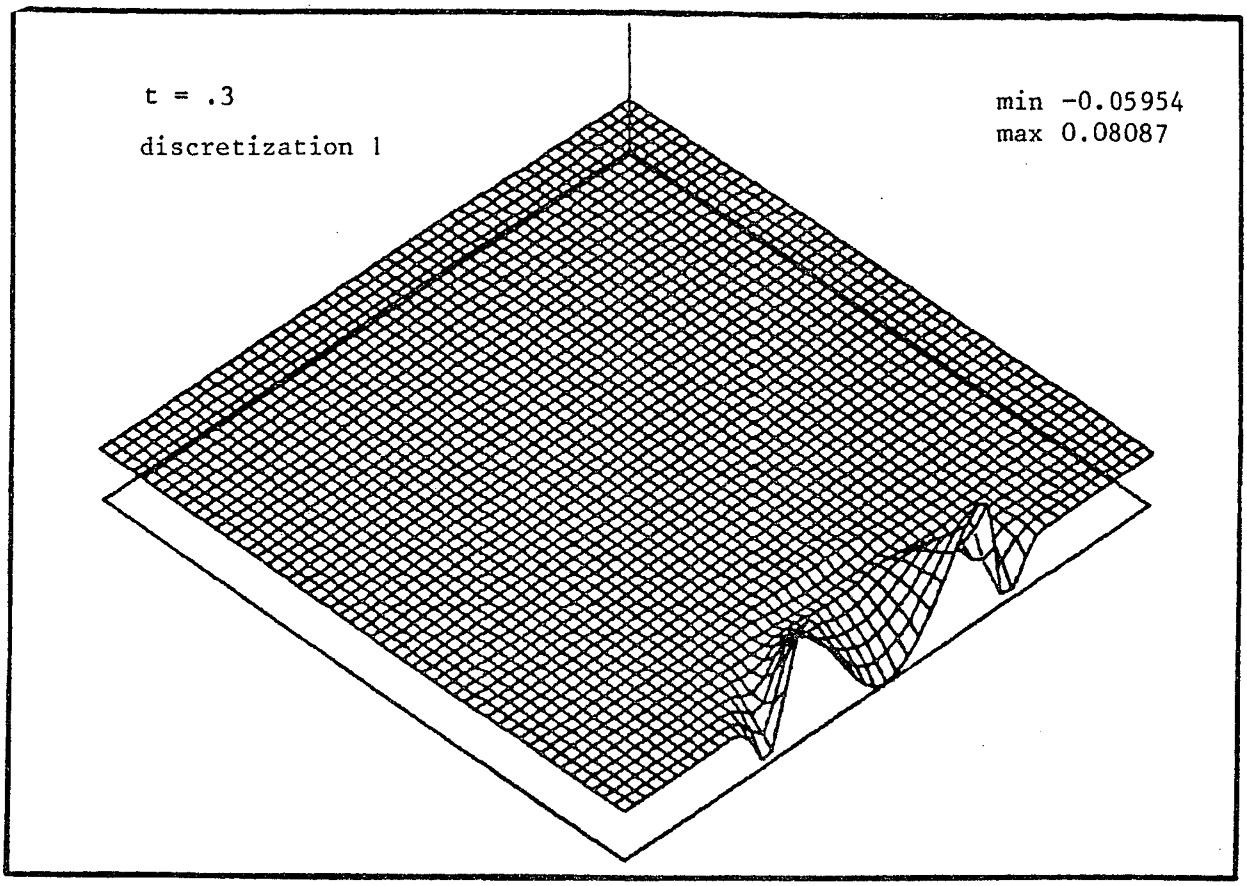


Figure 9 : difference between solution calculated with 2^{nd} order ABC and exact solutic

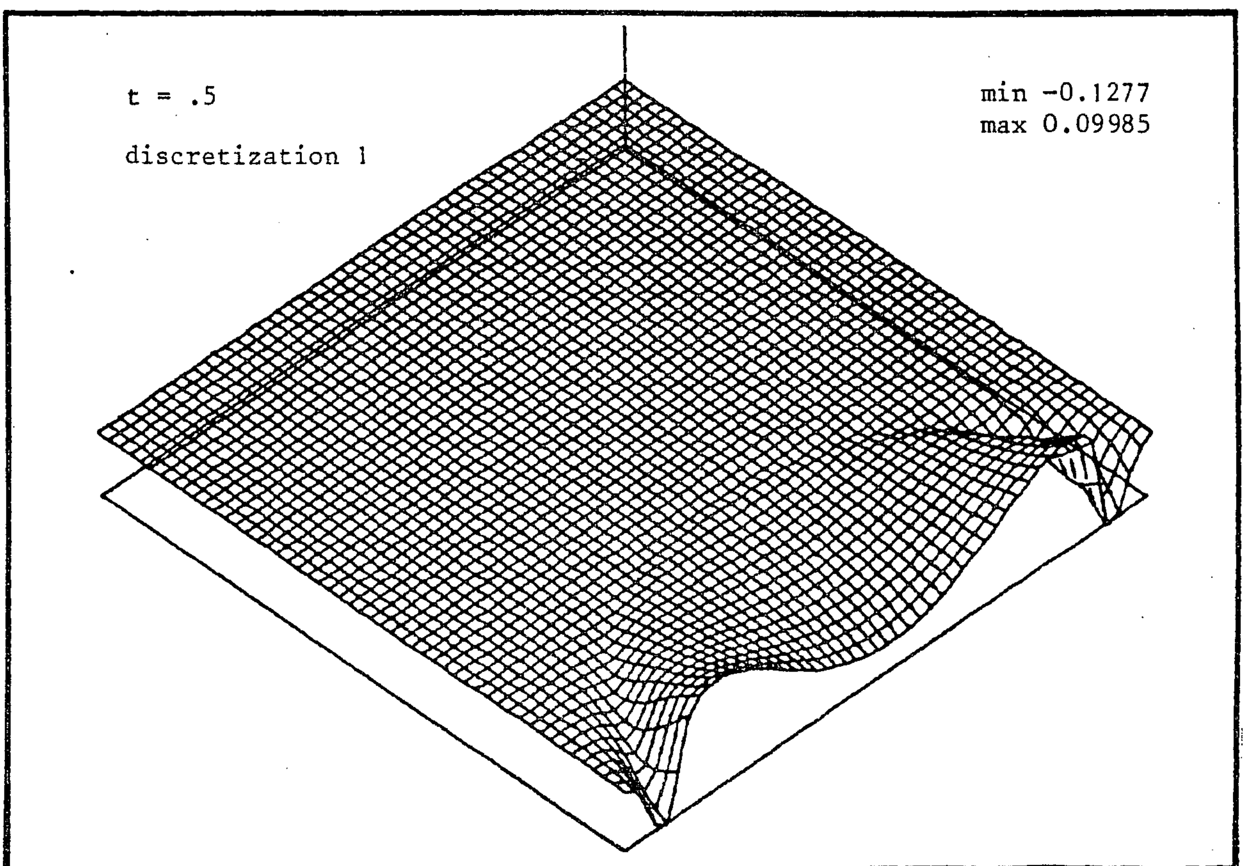


Figure 10: difference between solution calculated with 2^{nd} order ABC and exact solutior

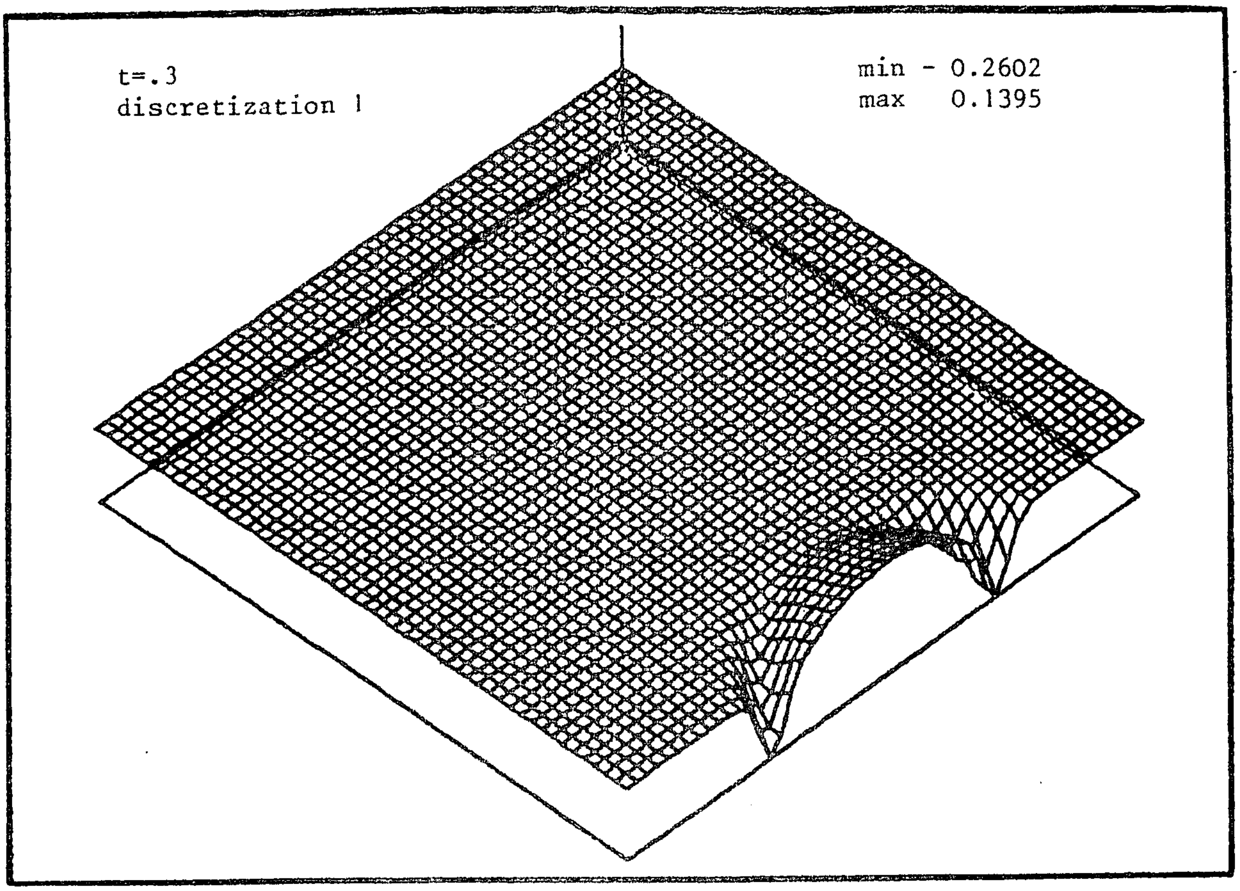


Figure 11: Difference between solution calculated with 1st order ABC and exact solution.

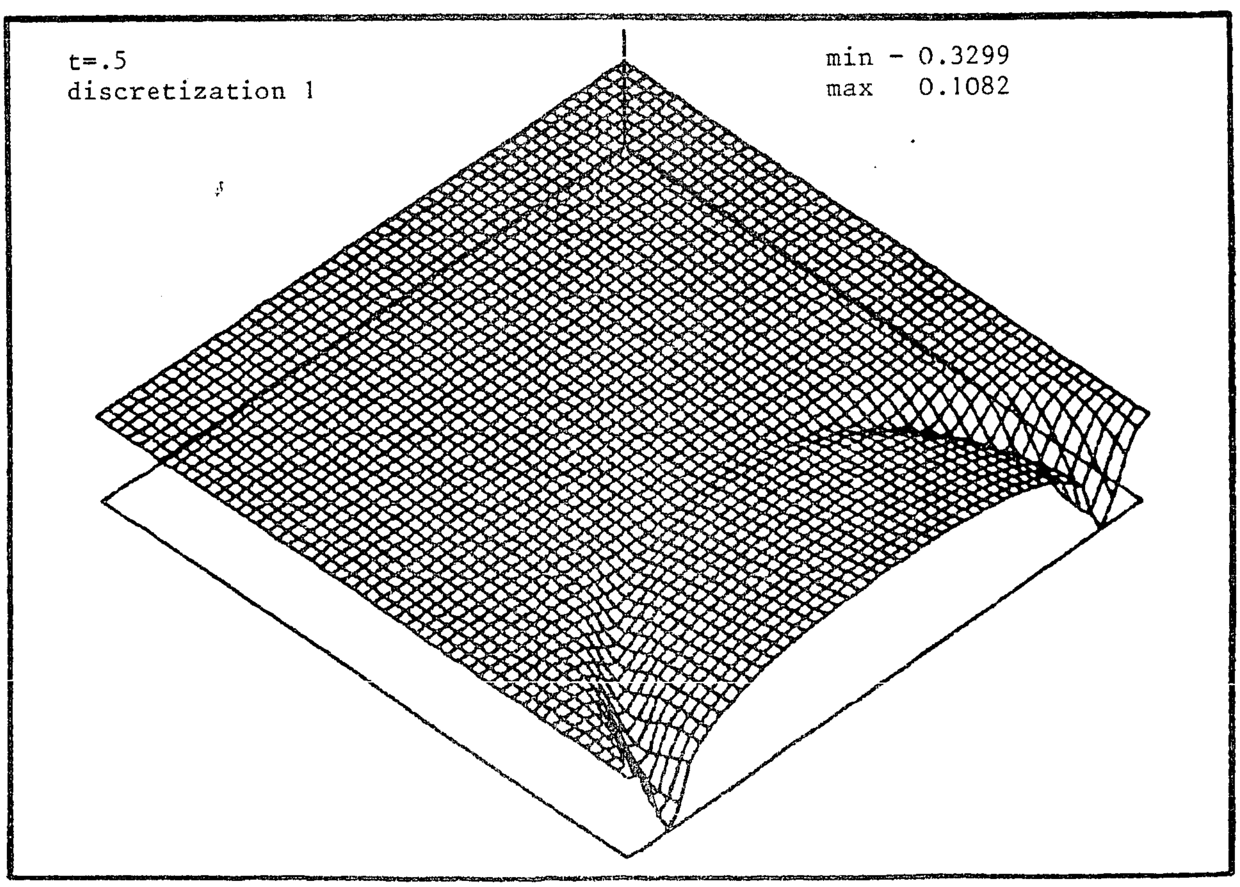


Figure 12: Difference between solution calculated with 1st order ABC and exact solution.

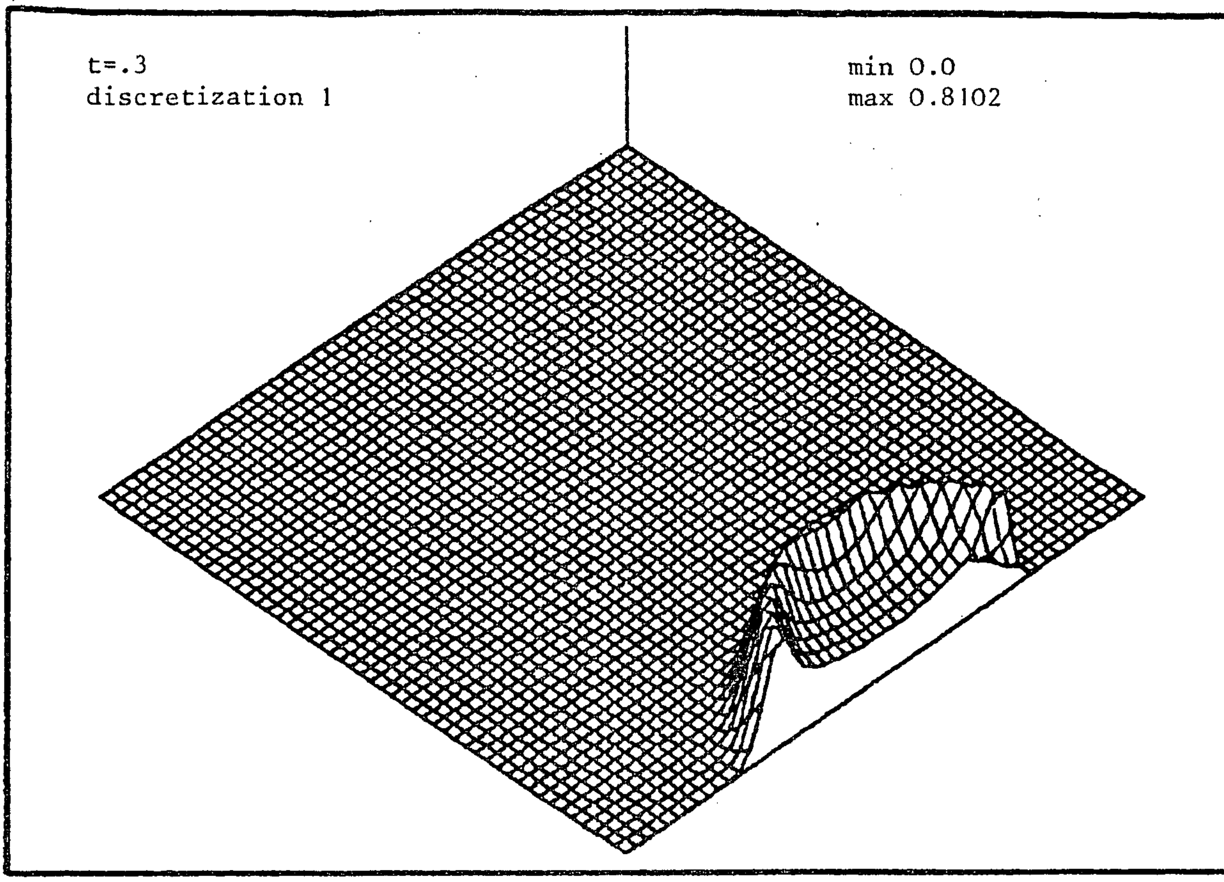


Figure 13 : Difference between solution calculated with Neumann BC and exact solution

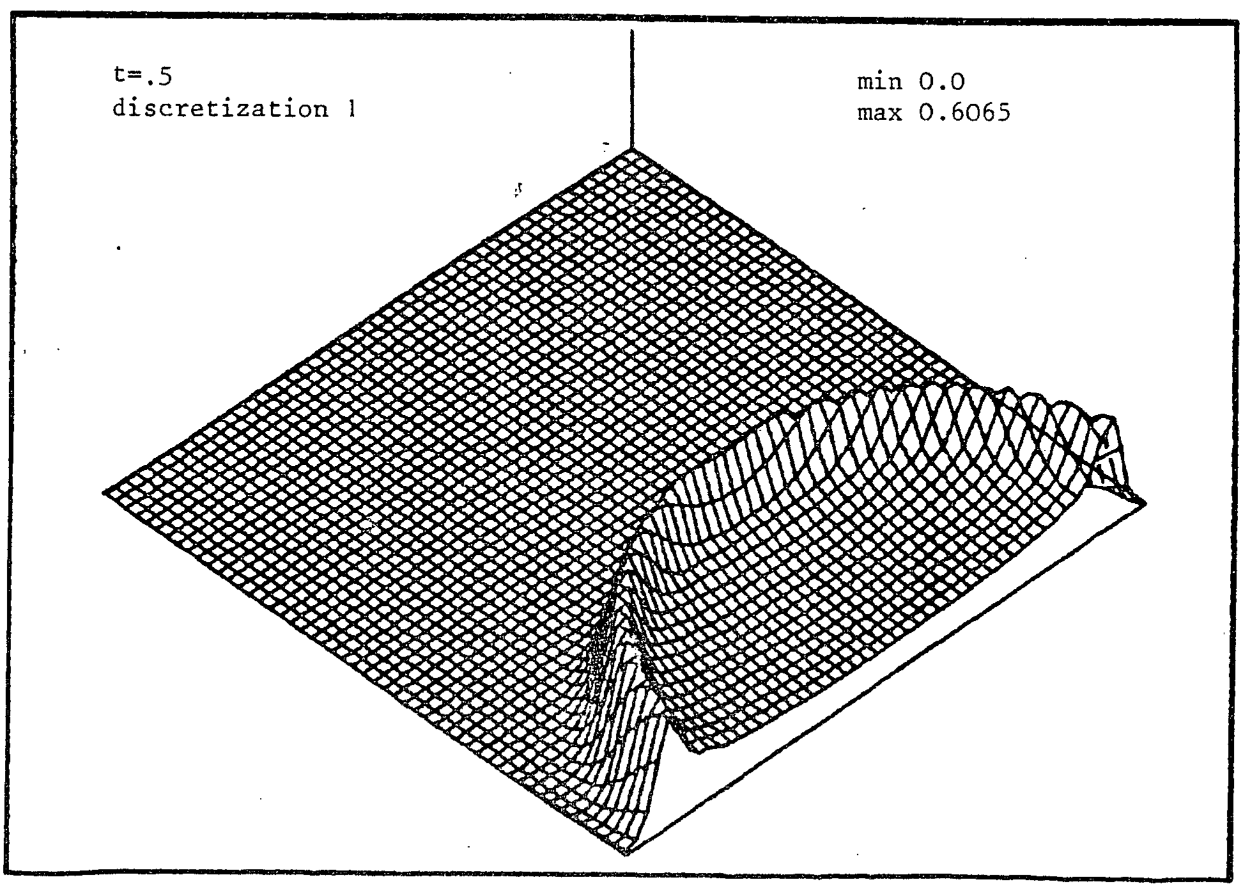


Figure 14: Difference between solution calculated with Neumann BC and exact solution

V.3.2 - Experiments concerning the corner condition

We turn now our attention to the results of the experiments concerned with the corner condition (5.3). Here we are primarily interested in comparing the results obtained with different choices of the constant γ though again we include the results of the same experiment with the 1st order condition and with the Neumann condition where, as remarked before, no special condition for the corner is needed.

In figures 15 and 16 we see the "exact" solution at times $t=.3$ and $t=.5$. For figures 17 through 22 the solutions were calcuated using the second order condition for the upper and right hand sides and the corner condition (5.3) for $\gamma=1.5$, 0.1, and 3.3. For figures 23 and 24 we have used the first order condition for the upper and right hand-sides while in figures 25 and 26 we have used the Neumann condition.

As before, except in the case of the Neumann boundary all of the reflected waves are dominated in amplitude by the initial wave so that the minimum and maximum are the same for corresponding times and the scale is the same in figures 15, 17, 19, 21, and 23 where $t=.3$ and again in figures 16, 18, 20, 22 and 24 where $t=.5$.

If we compare figures 17 and 18, as representative of the second order condition, and then 23 and 24, where the first order condition is used, with the exact solutions in 15 and 16 the superiority of the second order condition is perhaps more striking than in the earlier experiments. Still, as before, the difference is more pronounced in the figures where only the reflected wave obtained by subtracting the "exact" solution from the solution computed with the boundary condition is shown. These are figures 35 and 36 for the second order condition and figures 41 and 42 for the first order condition. The reflected wave for the Neumann condition is shown in figures 43 and 44. Once more we emphasize the difference in scale.

To compare the corner condition with different γ's we examine figures 17 and 18 for $\gamma=1.5$ (the "good" γ), figures 19 and 20 for $\gamma=0.1$, and figures 21 and 22 for $\gamma=3.0$. We note that in figures 19 and 20 and in 21 and 22 there can be seen a second wave following the initial wave that appears to be emitted by the corner. This is the γ wave or the corner wave which is seen again in figures 27 and 28 and in 29 and 30 where the differences between the solution for $\gamma=1.5$ and those for $\gamma=0.1$ and $\gamma=3.0$ are given.

To see that the γ-wave is not a discretization phenomena that tends to 0 with Δt and Δx, we have depicted in figures 31 through 34 the same waves as in figures 27 through 30 respectively but now obtained with a discretization in which Δx and Δt have been halved, Δx=.005, Δt=.0025 (discretization 2).

We remark that in figures 17 and 18 where the "good" γ, γ=1.5, has been used, there still seems to be a second wave following the initial wave, but much smaller than that seen in figures 19 through 22. This is in fact, however, only the effect of the 2nd order absorbing boundary condition, cf. figures 9 and 10.

The differences between the solutions for the three choices of γ and the "exact" solution are shown in figures 35 through 40. Besides the γ waves we see in each of these figures a disturbance further from the corner on both the upper and right hand edges. This corresponds to the reflection due to the second order absorbing boundary condition transparent only for waves at normal incidence as we have seen in figures 9 and 10.

The last 12 figures, 45 through 56, are the seismograms giving the response at six points near the corner. For the odd numbered figures, those shown on the left hand side of the page, the solid line represents the exact solution, the dotted line the solution obtained with the second order condition with γ=1.5, and the dashed line the solution obtained with the first order condition. At all of the points, even at point 1 which is farthest from the boundary, the better behavior of the second order condition is quite pronounced.

For the even numbered figures the three curves all represent solutions obtained with the second order condition but with different choices of the constant γ. For the solution represented by the solid line γ=1.5, by the dotted line γ=0.1, and by the dashed line γ=3.0. Again at each point it is quite clear, especially on comparing with the exact solution in the corresponding odd numbered figure, that the better solution is obtained with γ=1.5. We do remark that by the final time of the observation that all of the solutions with the second order condition have nearly converged to the exact solution while the solution with the first order condition remains at some distance.

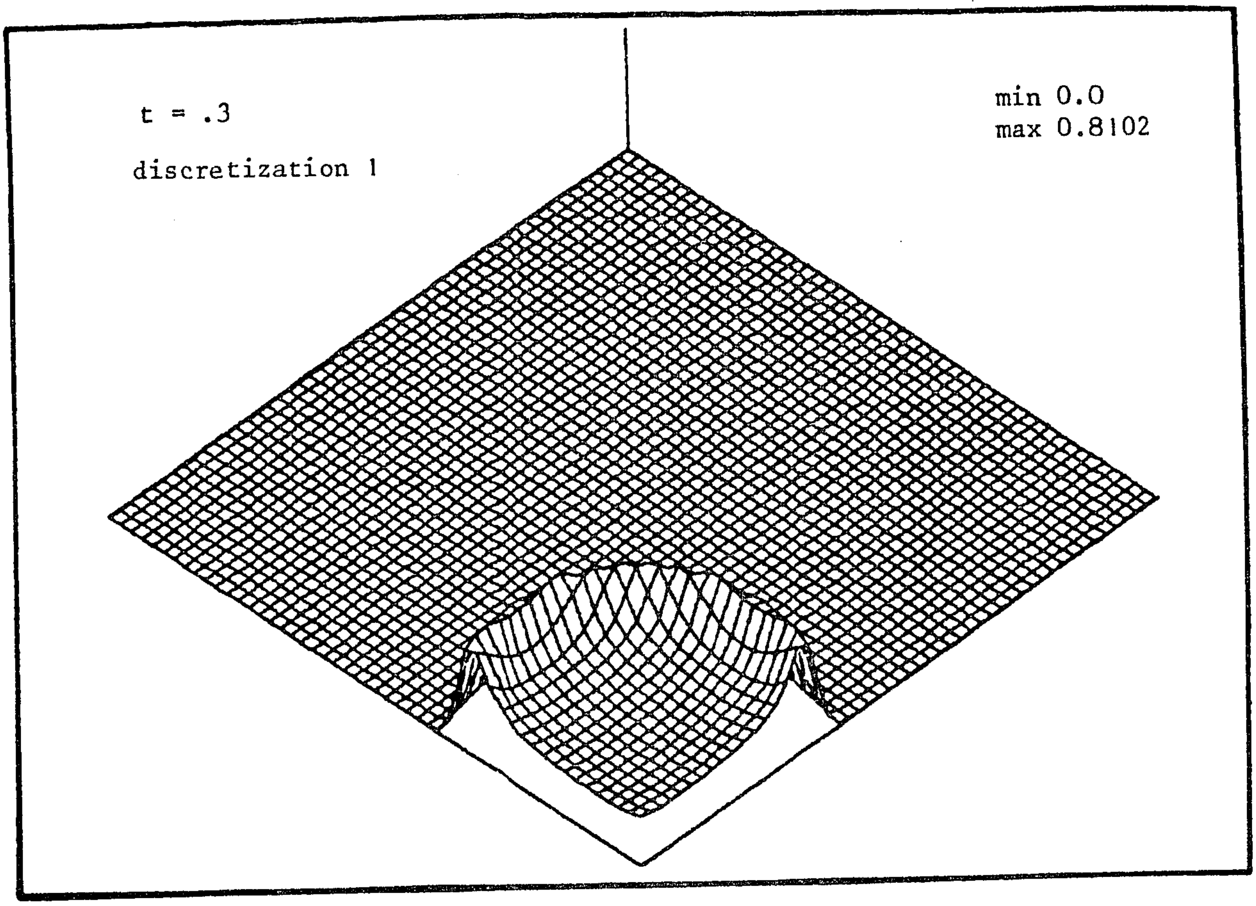


Figure 15: exact solution

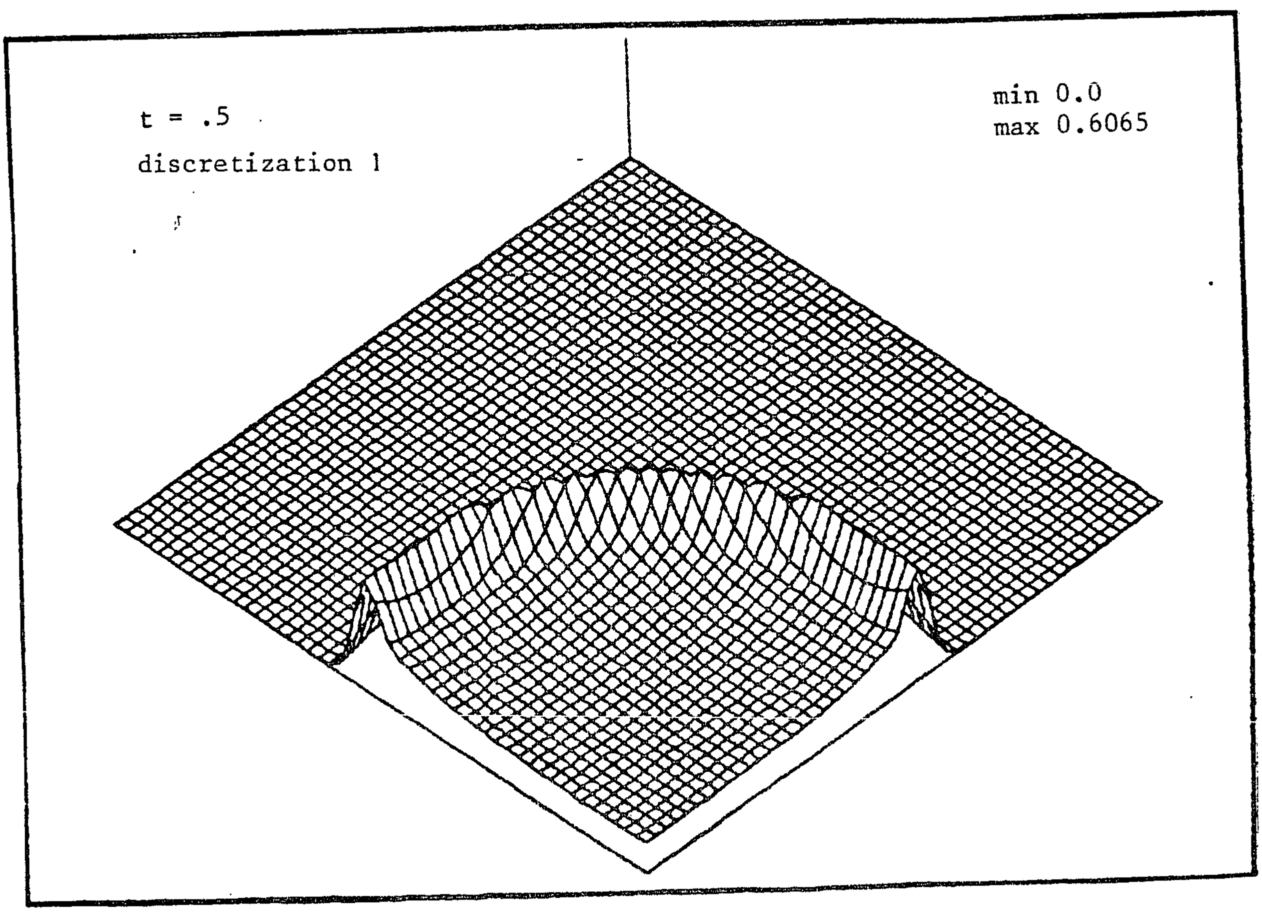


Figure 16: exact solution

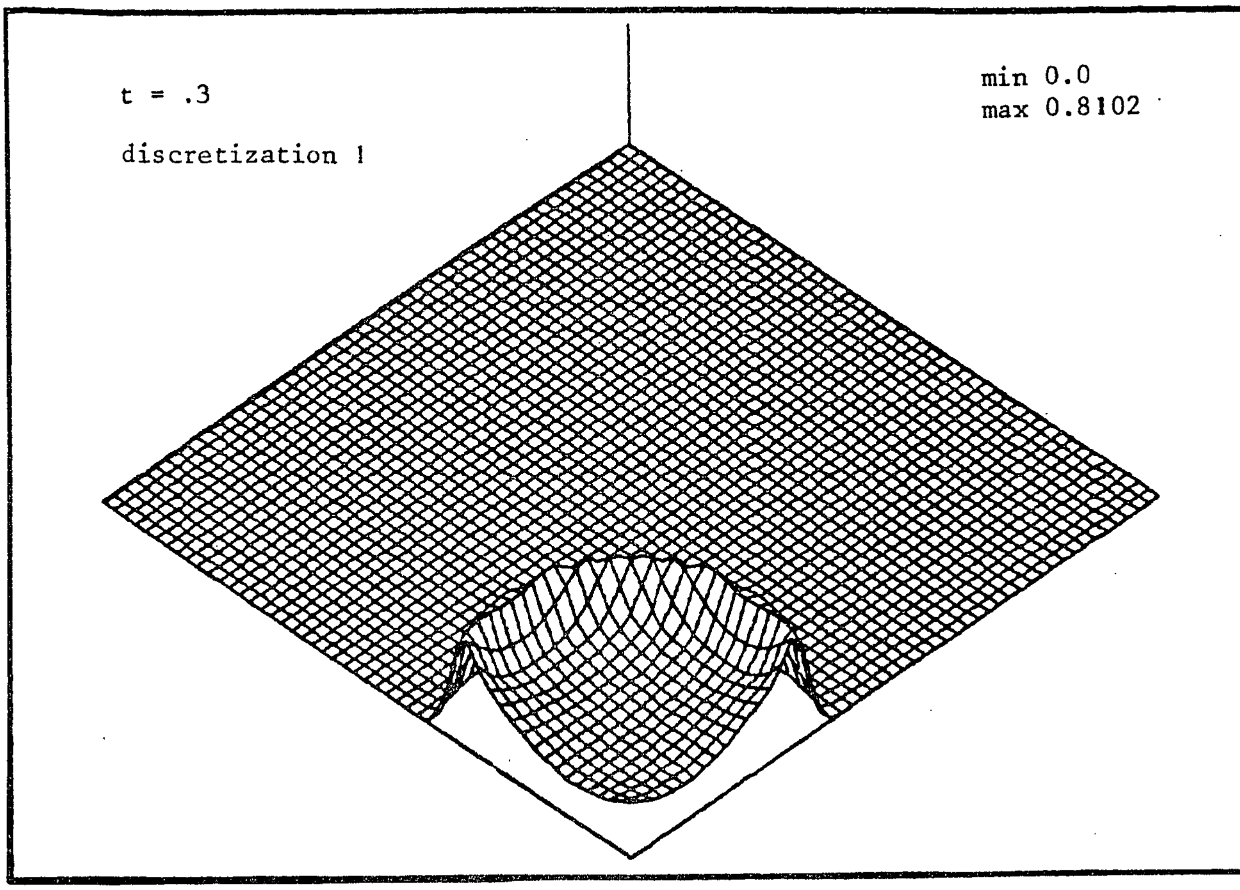


Figure 17: solution calculated with γ=1.5

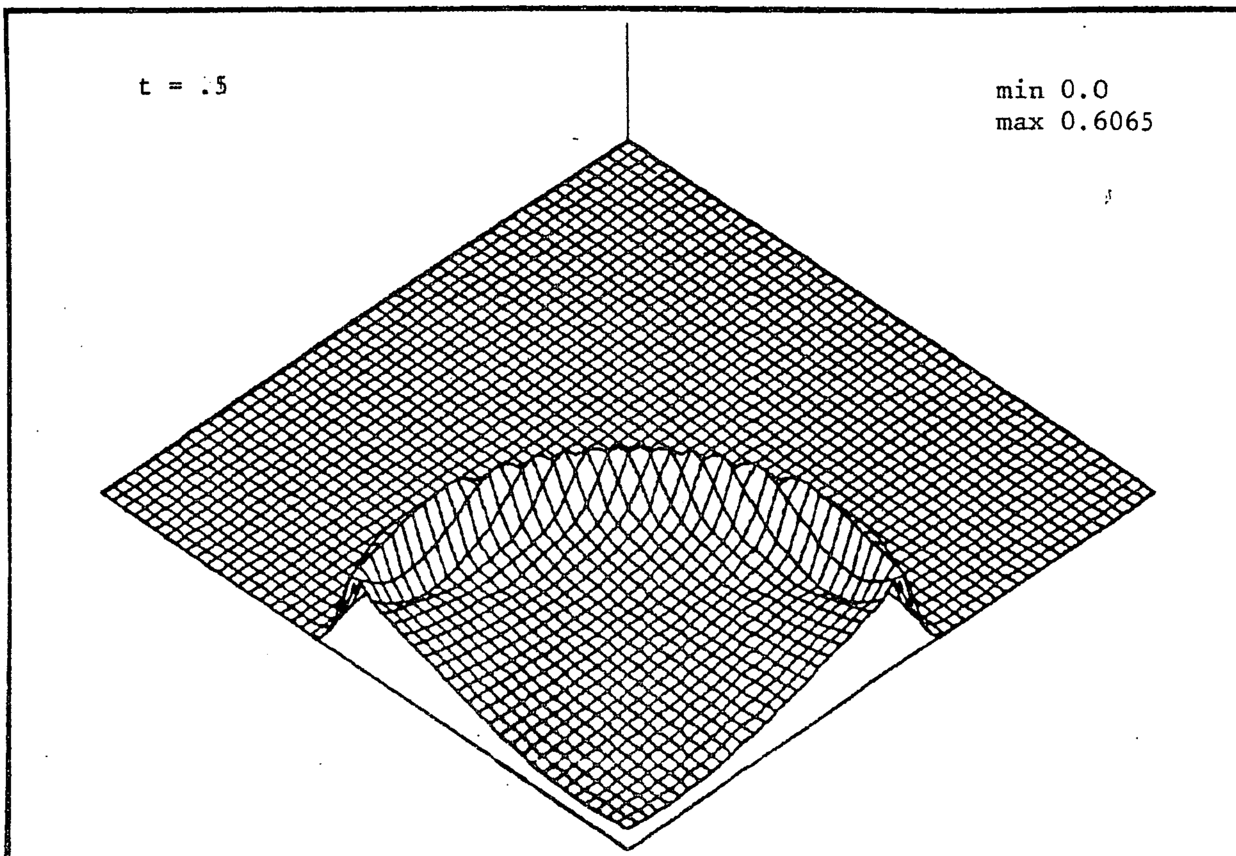


Figure 18 : solution calculated with γ=1.5

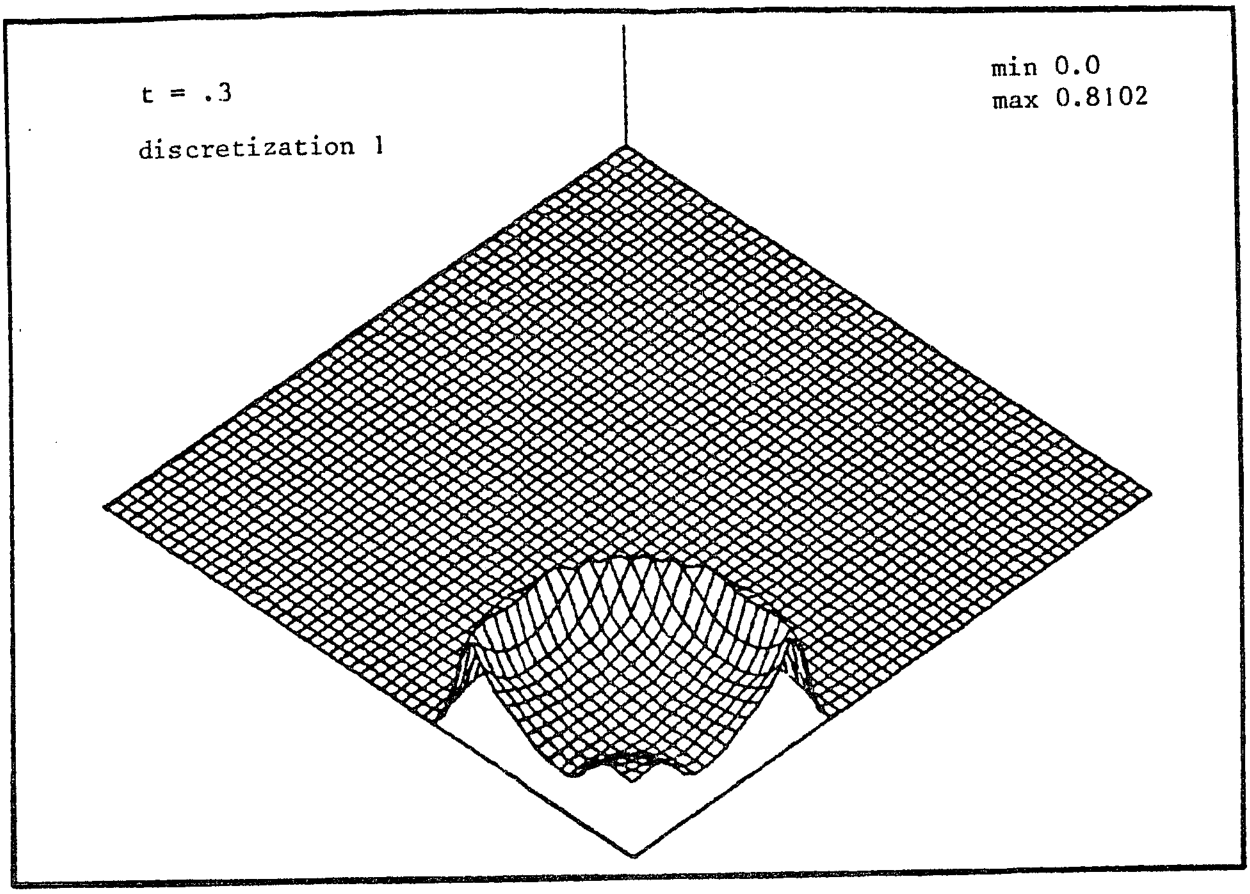


Figure 19 : solution calculated with $\gamma=0.1$

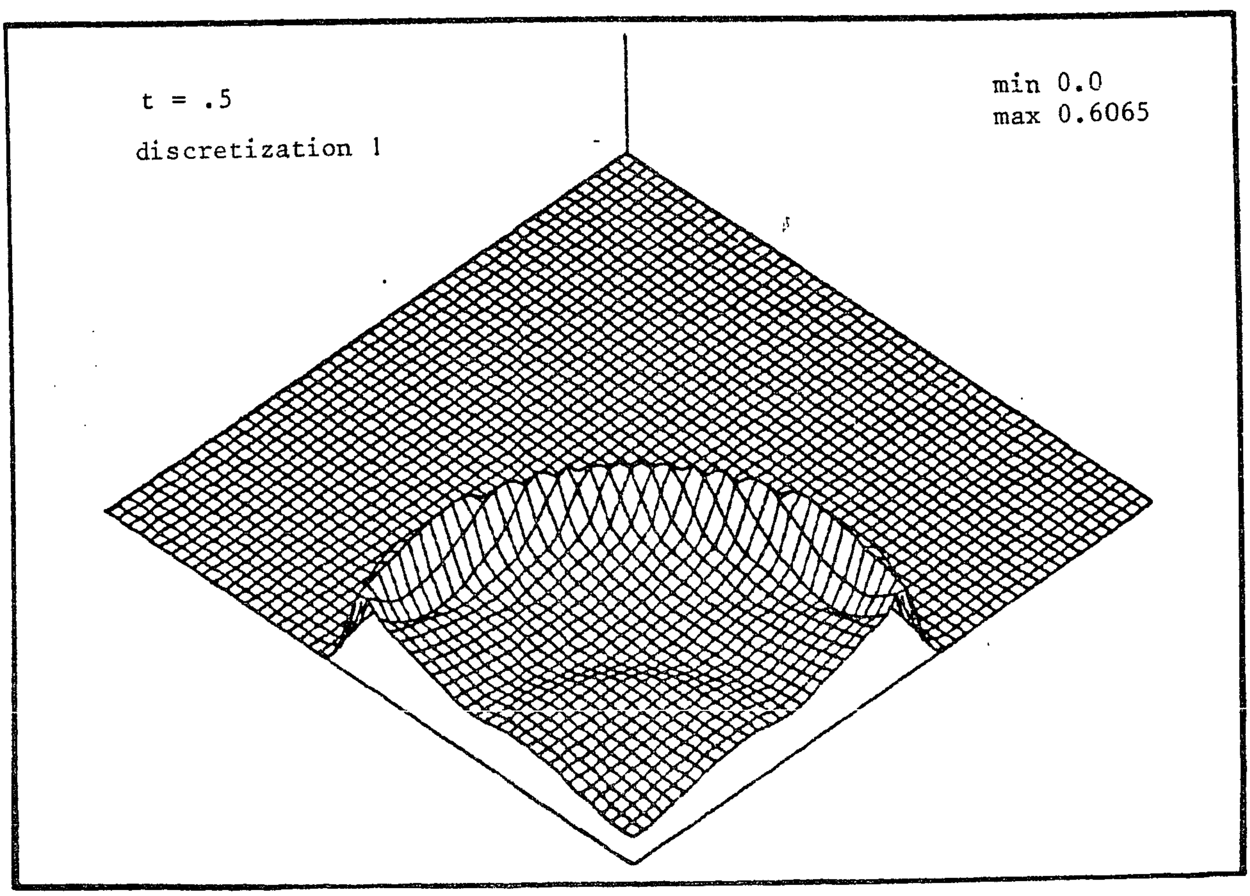


Figure 20 : solution calculated with $\gamma=0.1$

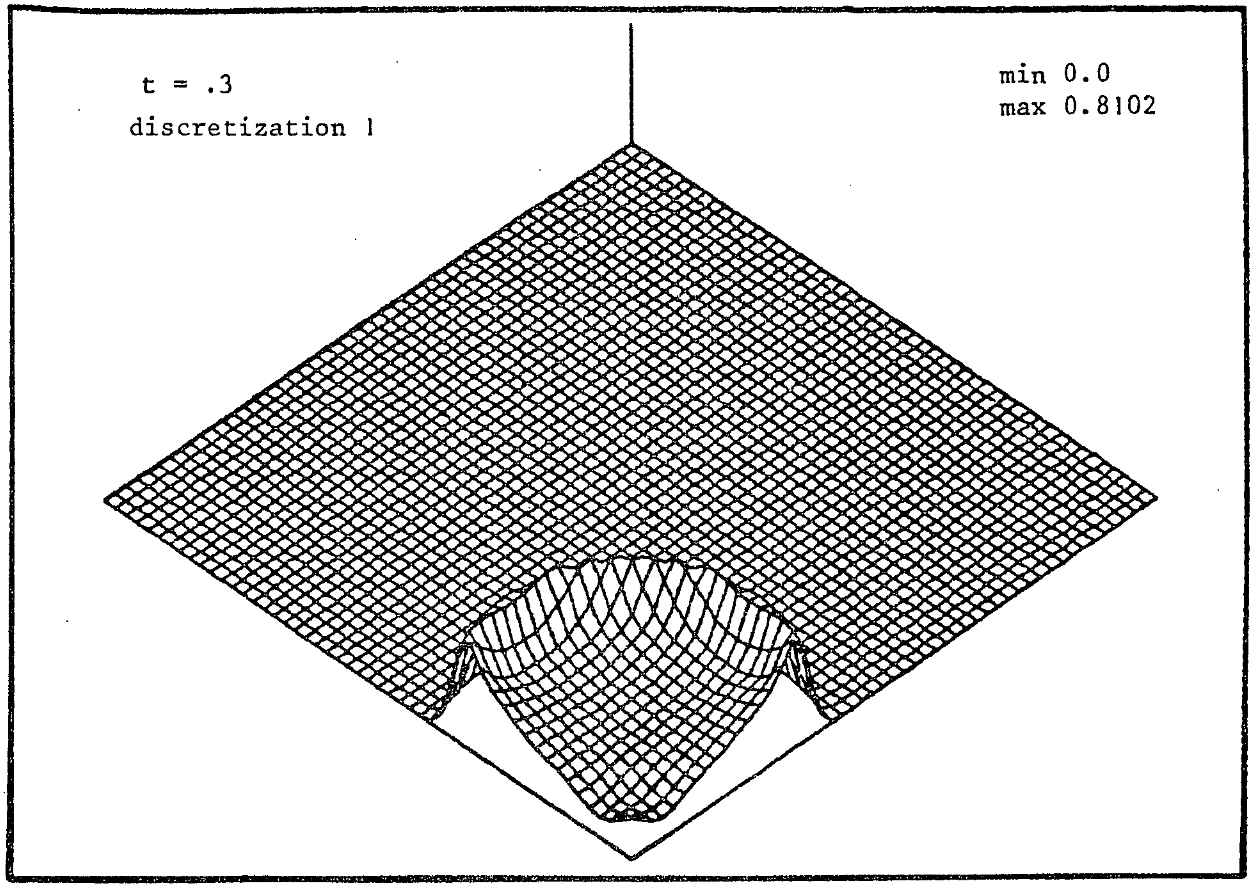


Figure 21 : solution calculated with $\gamma=3$

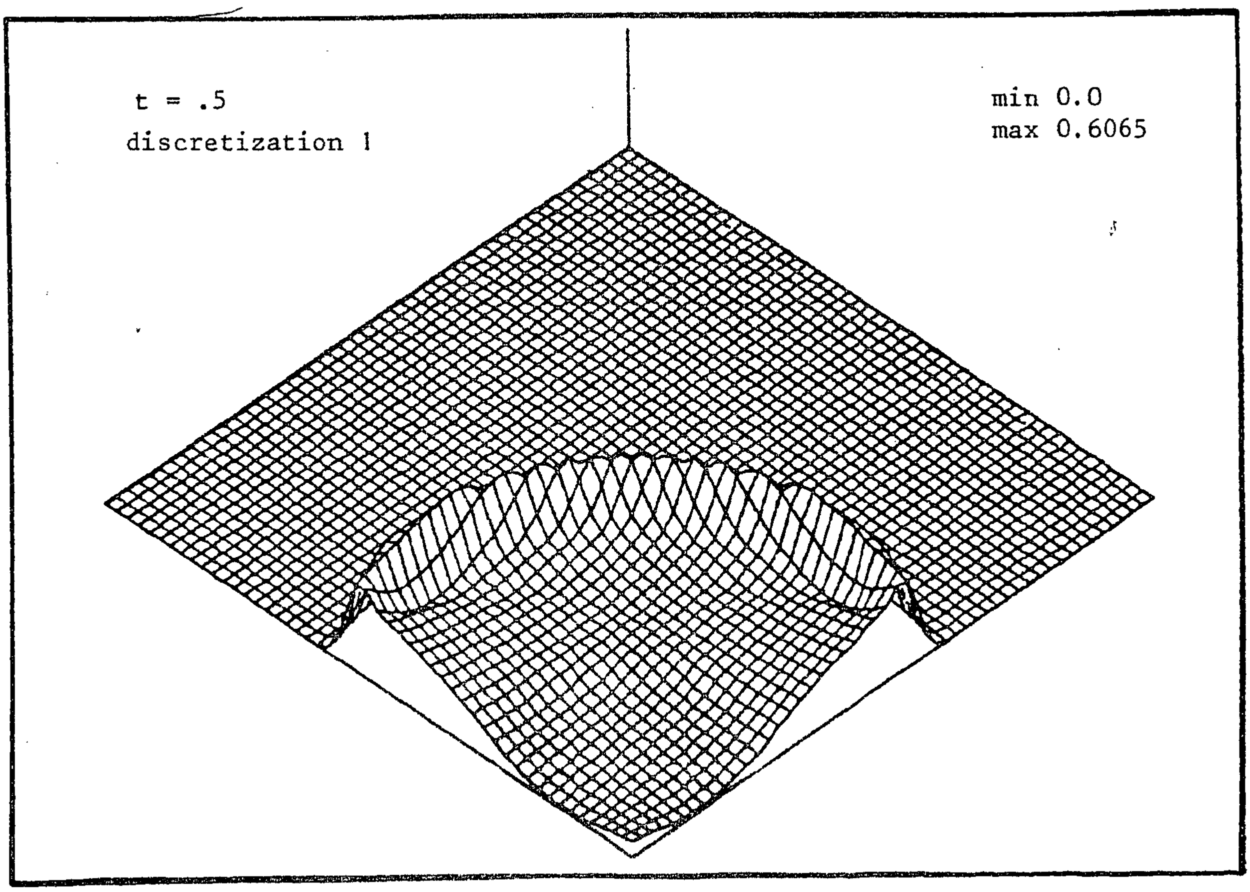


Figure 22 : solution calculated with $\gamma=3$

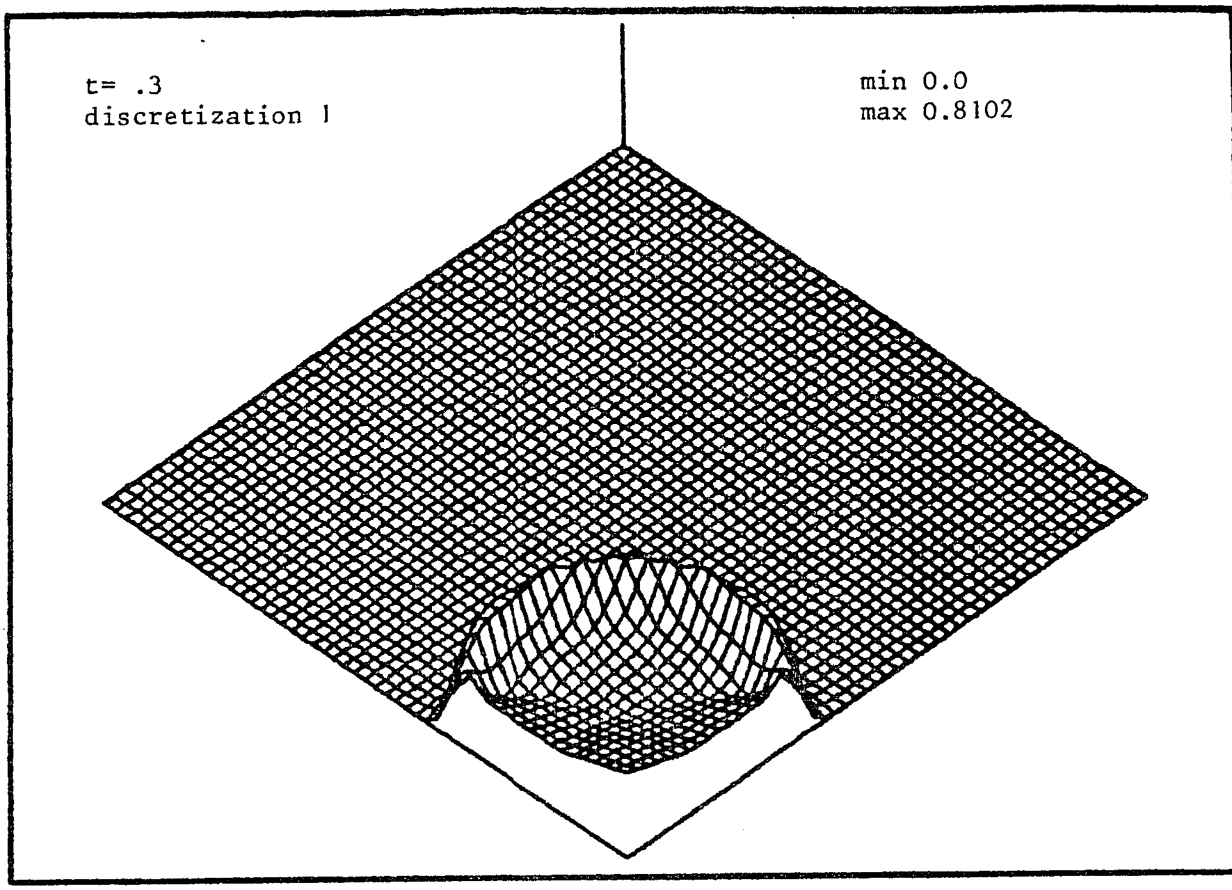


Figure 23: Solution calculated with 1st order ABC

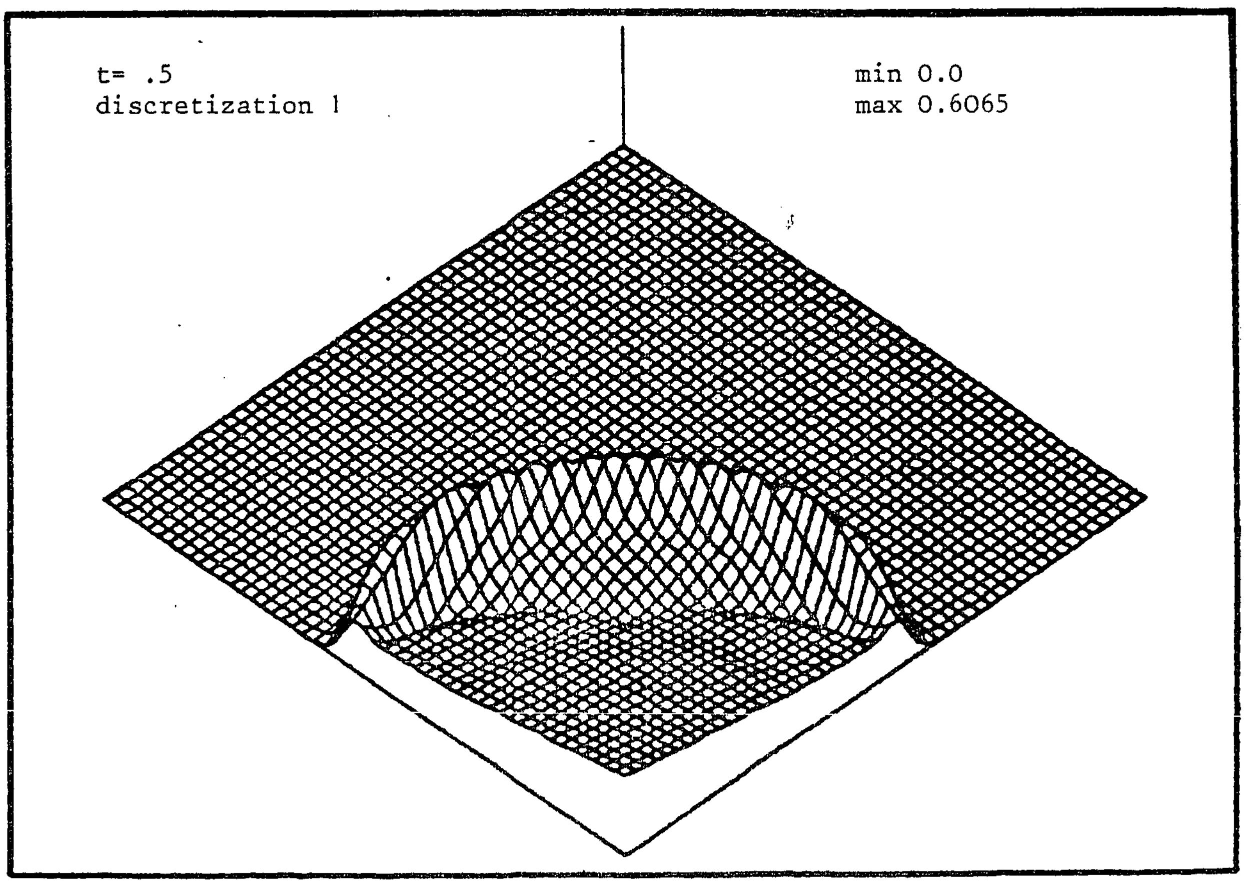


Figure 24: Solution calculated with 1st order ABC

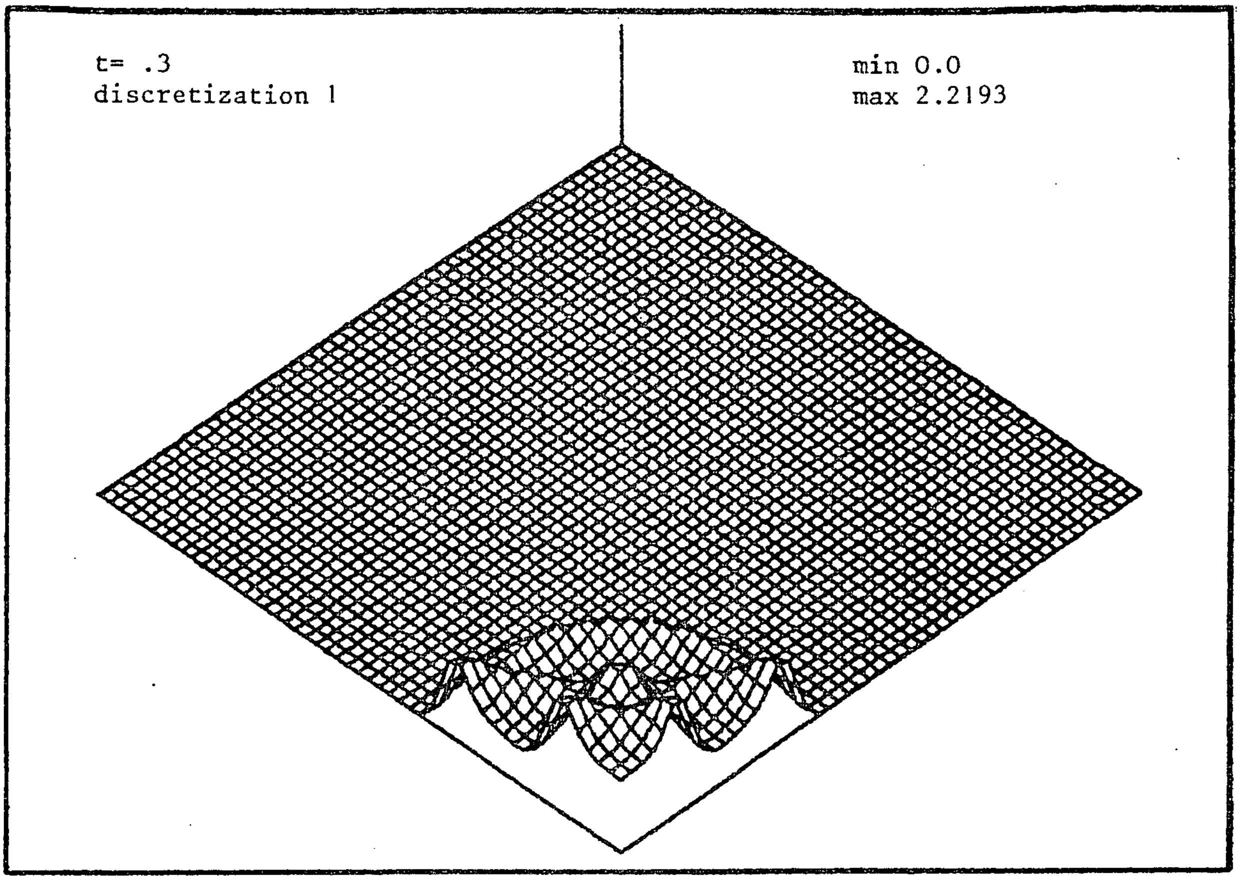


Figure 25: Solution calculated with Neumann BC

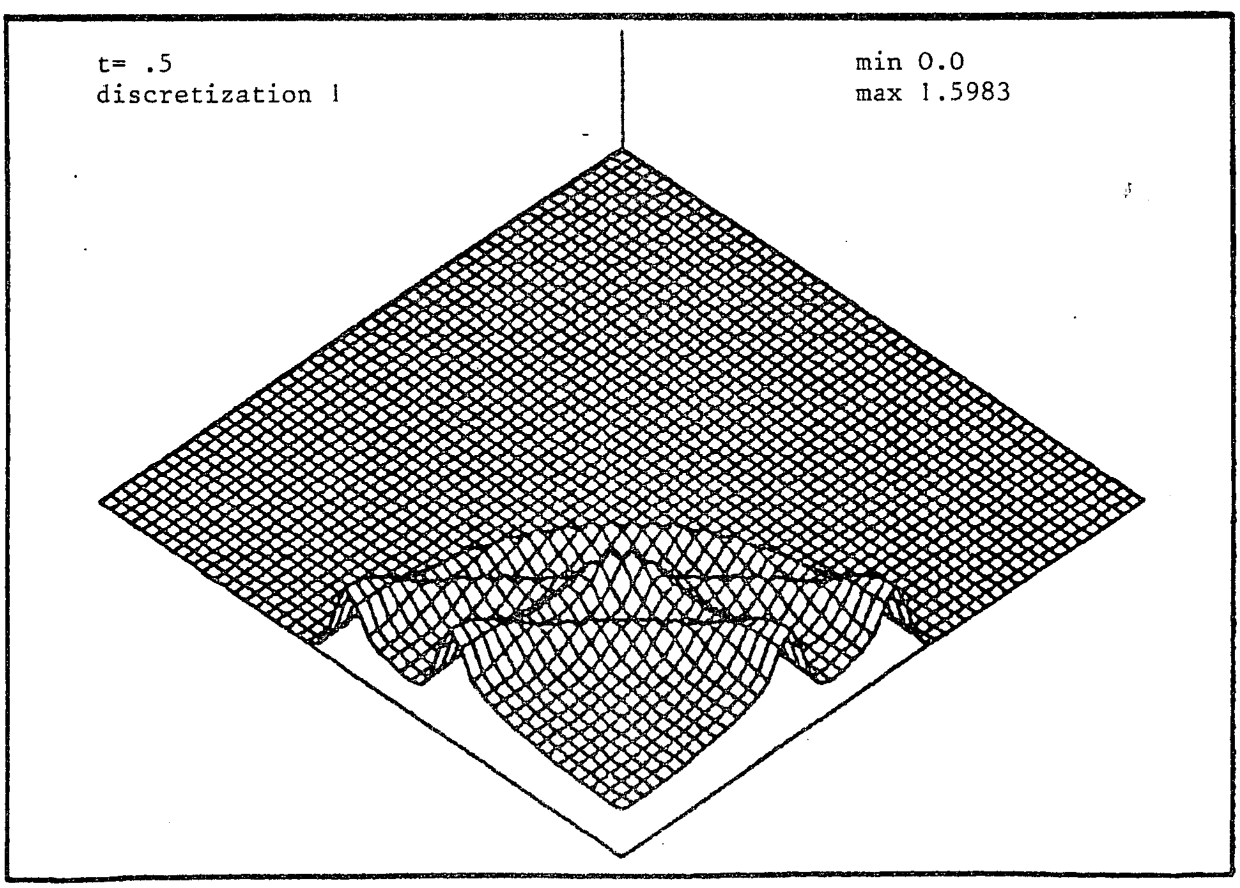


Figure 26: Solution calculated with Neumann BC

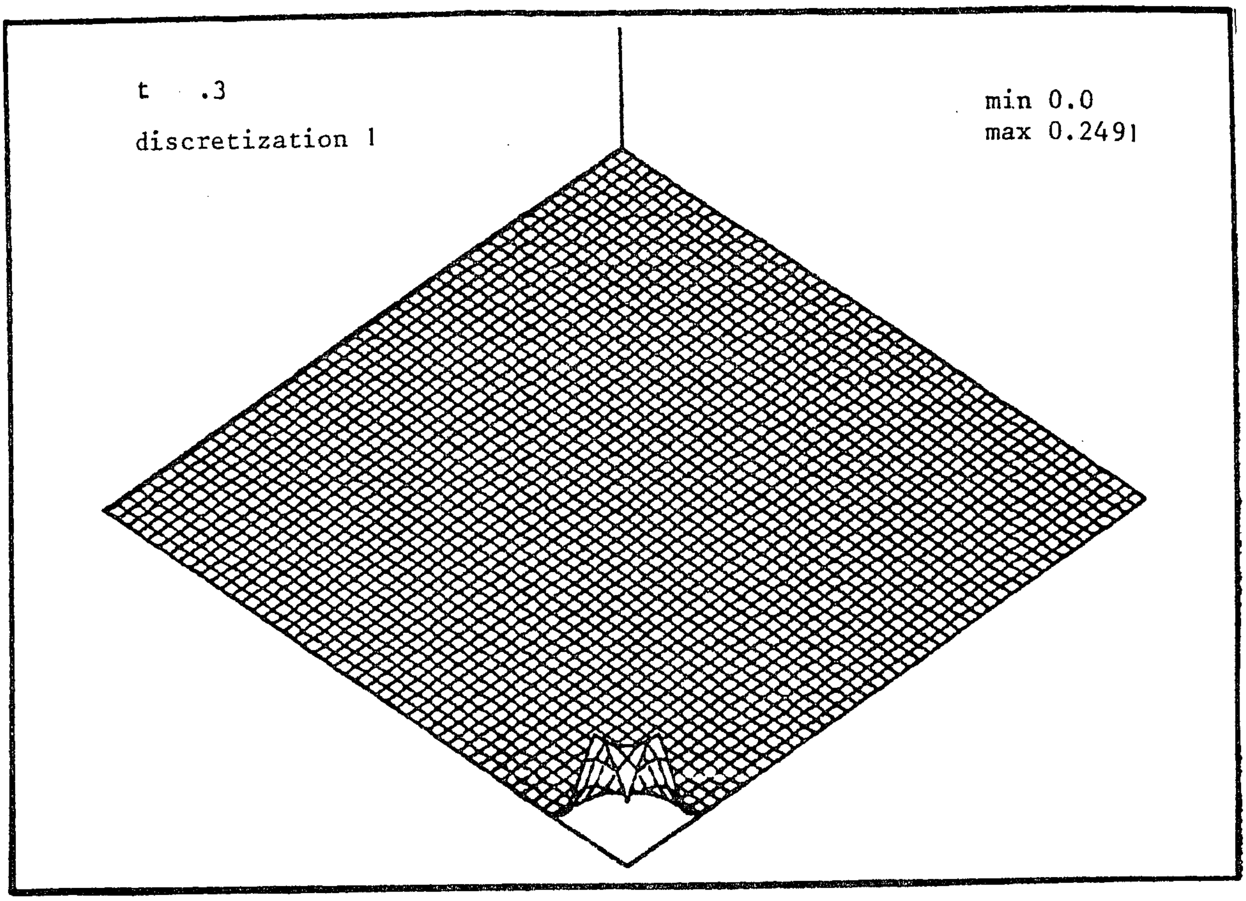


Figure 27 : difference between solutions calculated with γ=0.1 and with γ=1.5

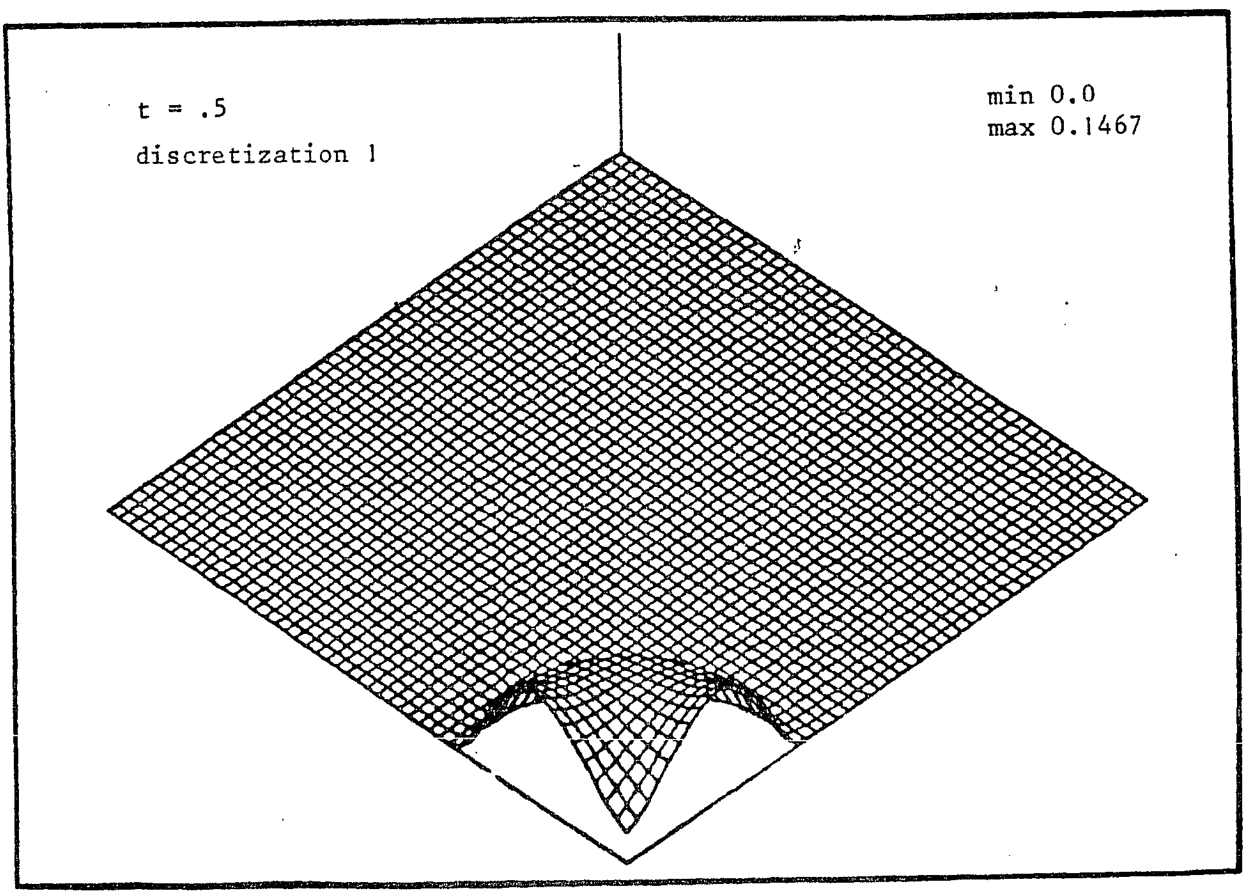


Figure 28 : difference between solutions calculated with γ=0.1 and γ=1.5

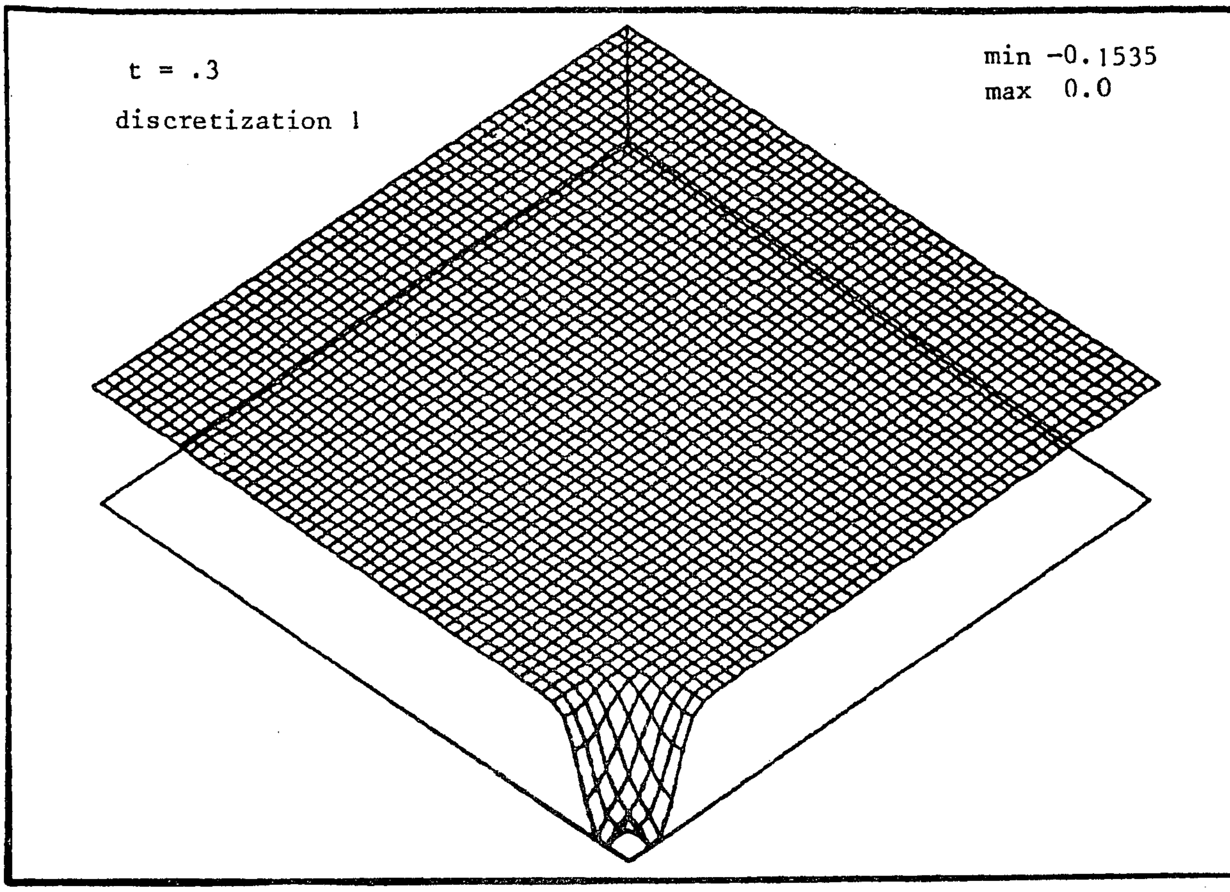


Figure 29 : difference between solutions calculated with $\gamma=3$ and with $\gamma=1.5$

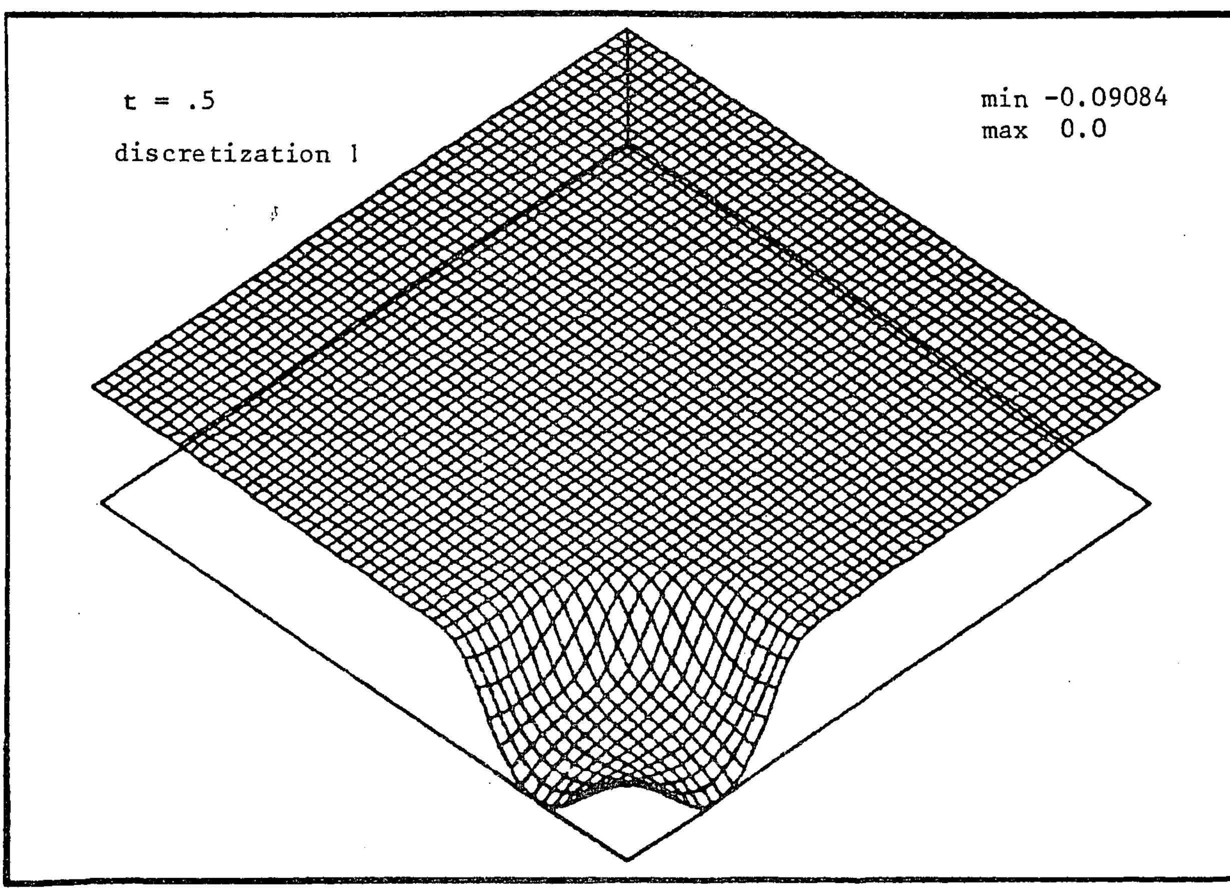


Figure 30 : difference between solutions calculated with $\gamma=3$ and with $\gamma=1.5$

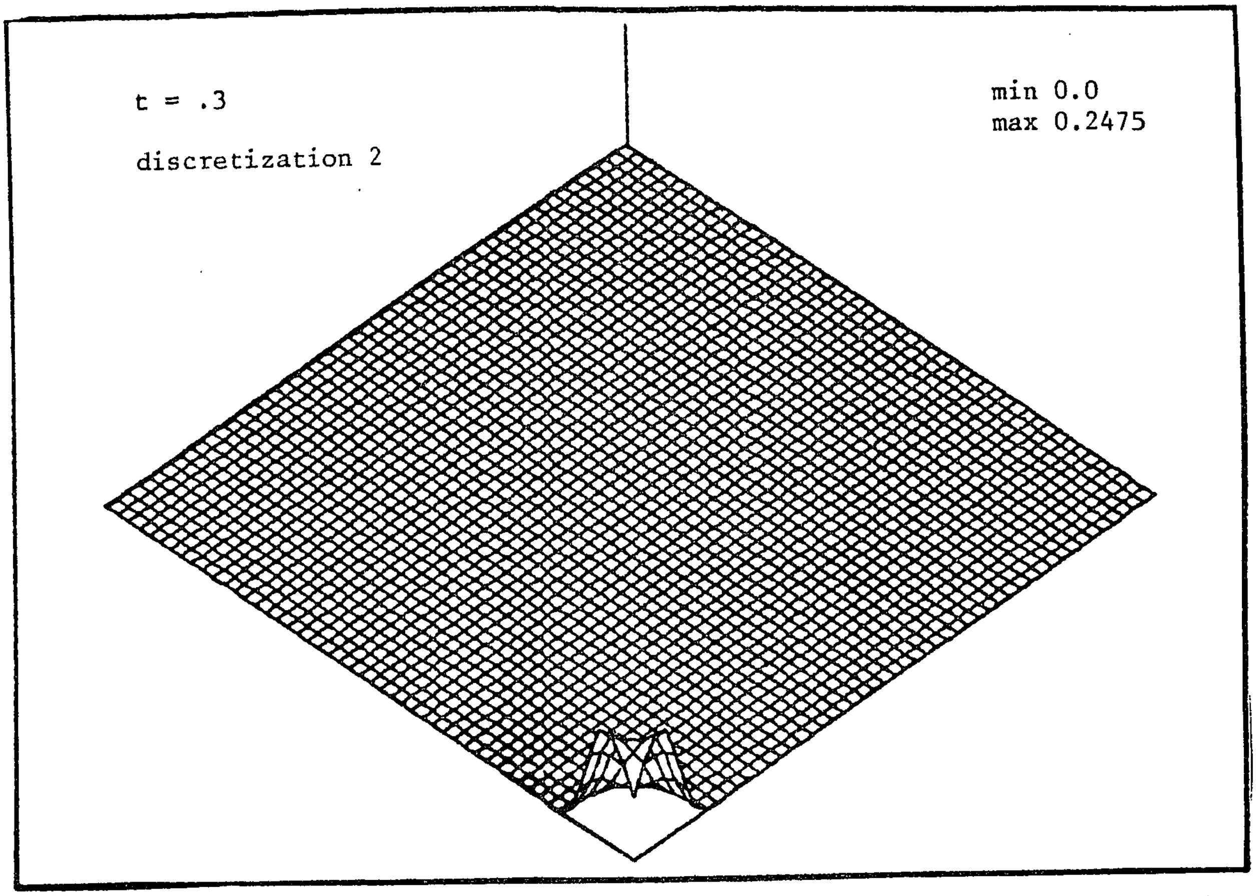


Figure 31 : difference between solutions calculated with $\gamma=0.1$ and with $\gamma=1.5$

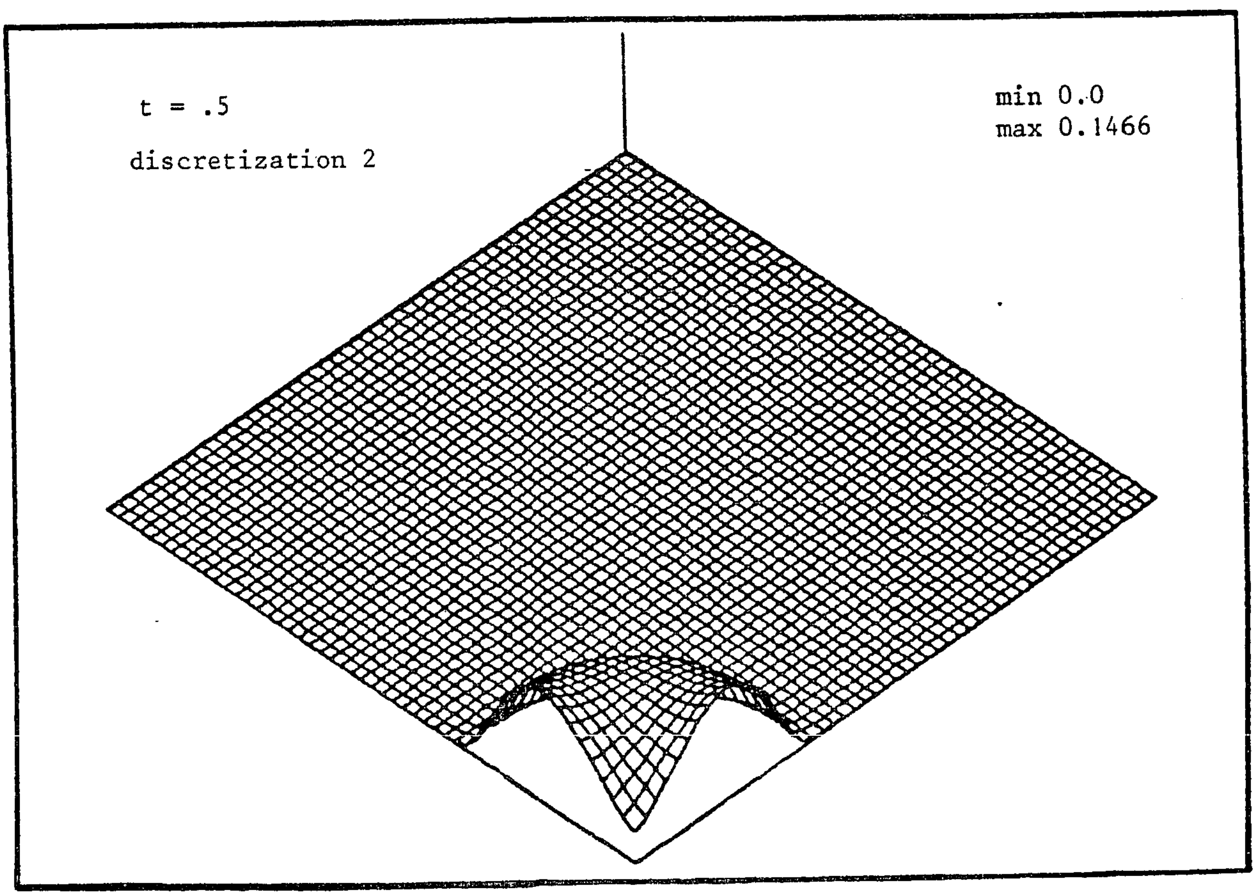


Figure 32 : difference between solutions calculated with $\gamma=0.1$ and with $\gamma=1.5$

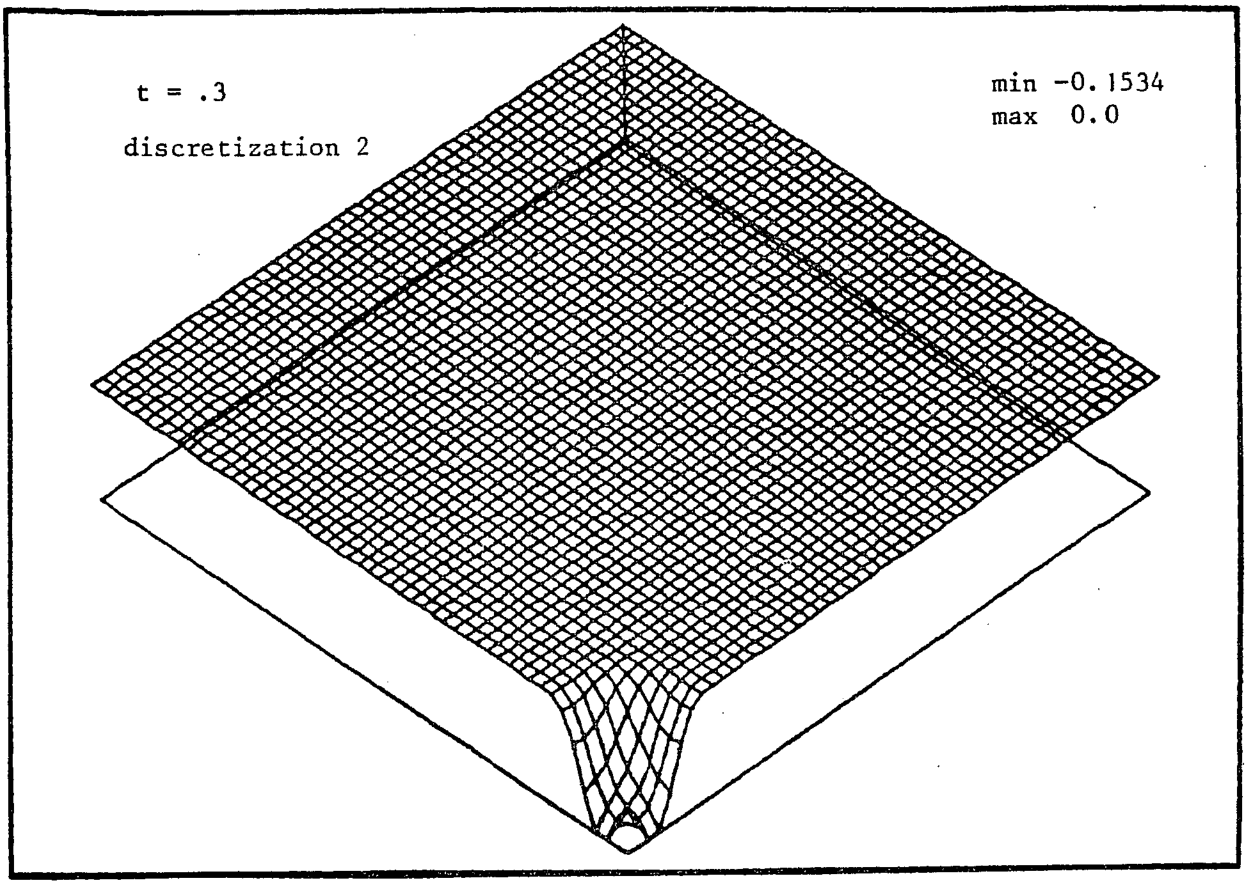


Figure 33 : difference between solutions calculated with γ=3. and with γ=1.5

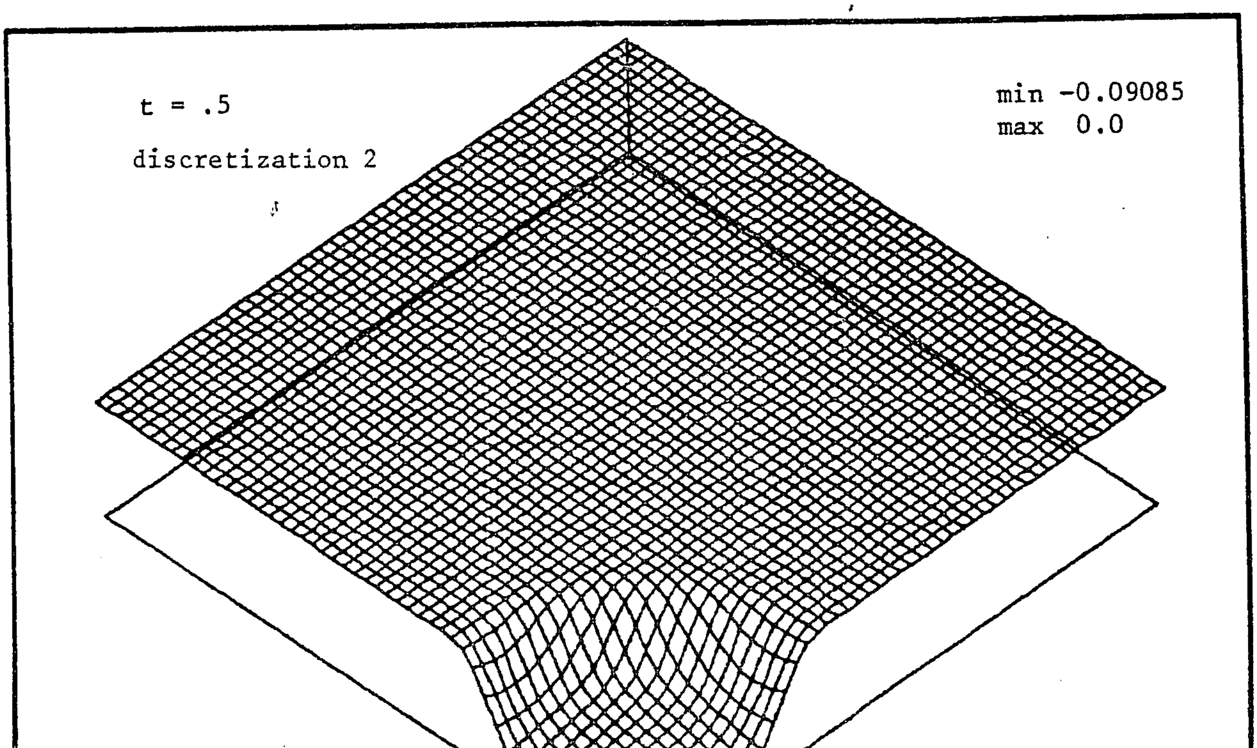


Figure 34 : difference between solutions calculated with γ=3. and with γ=1.5

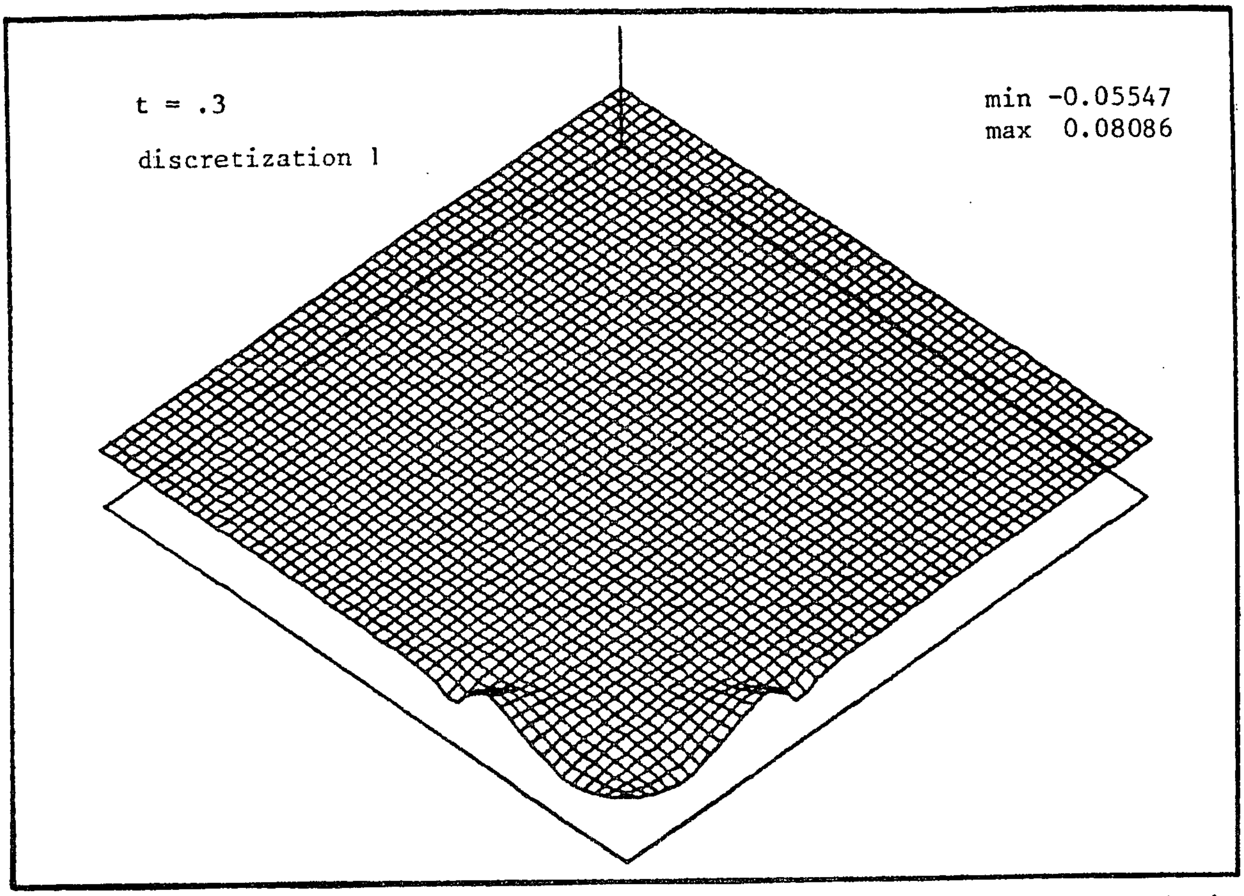


Figure 35 : difference between solution calculated with $\gamma=1.5$ and exact solution

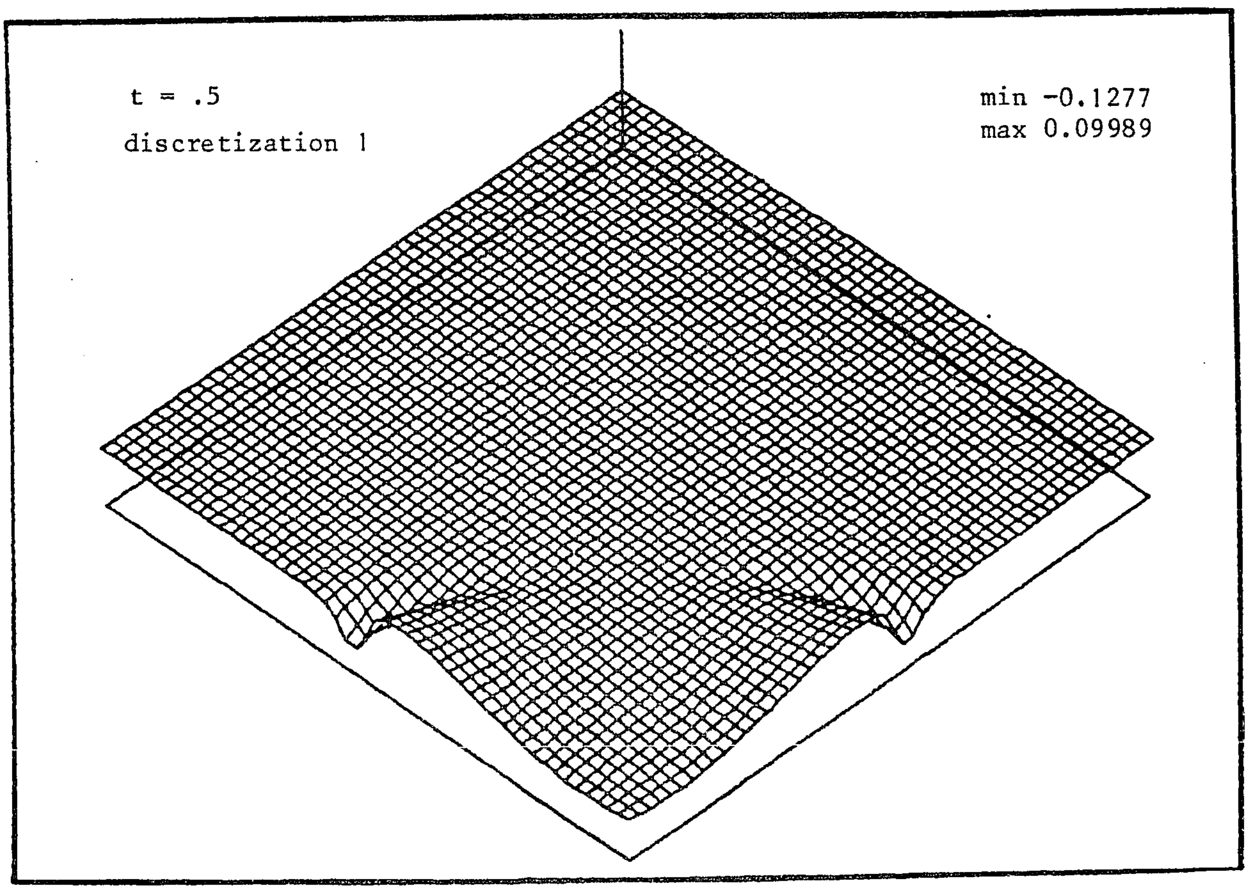


Figure 36 : difference between solution calculated with $\gamma=1.5$ and exact solution

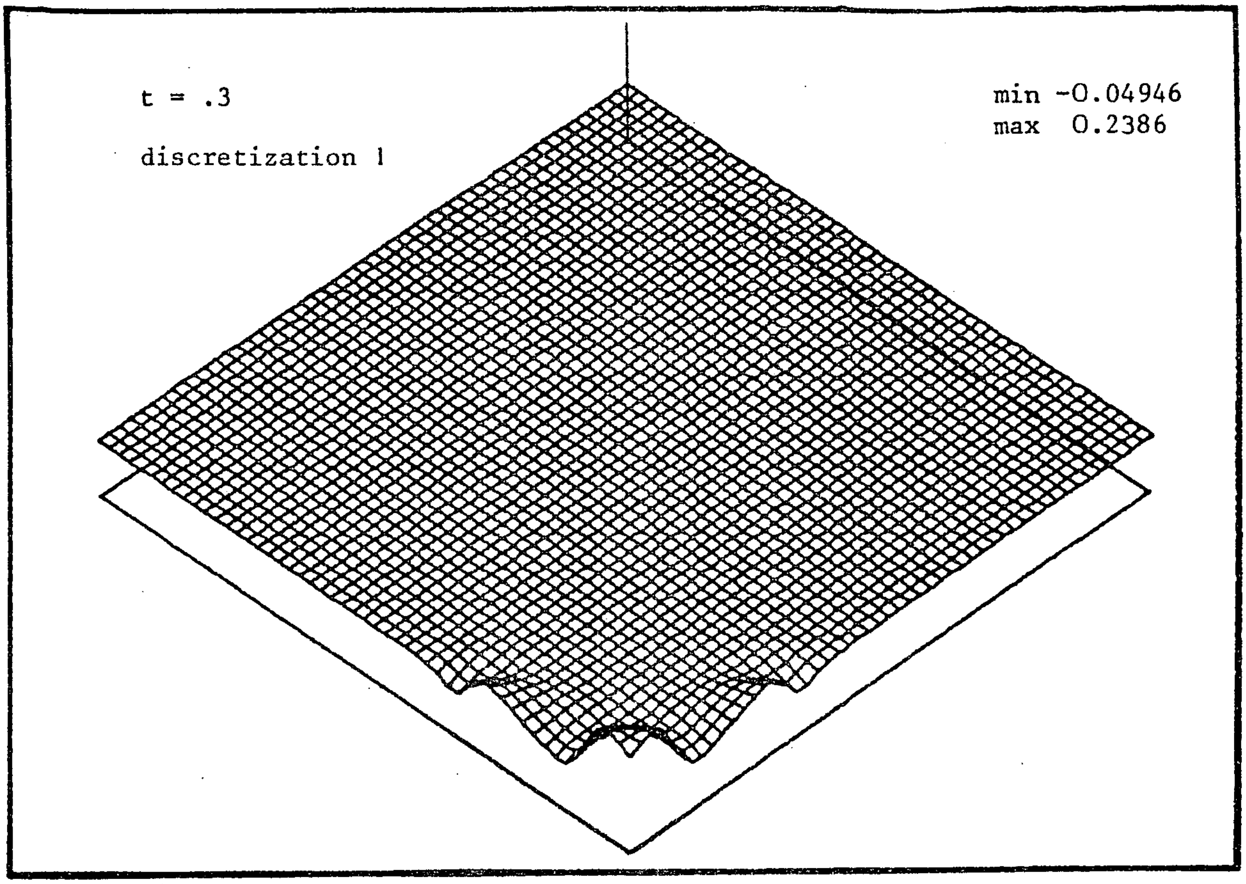


Figure 37 : difference between solution calculated with γ=0.1 and exact solution

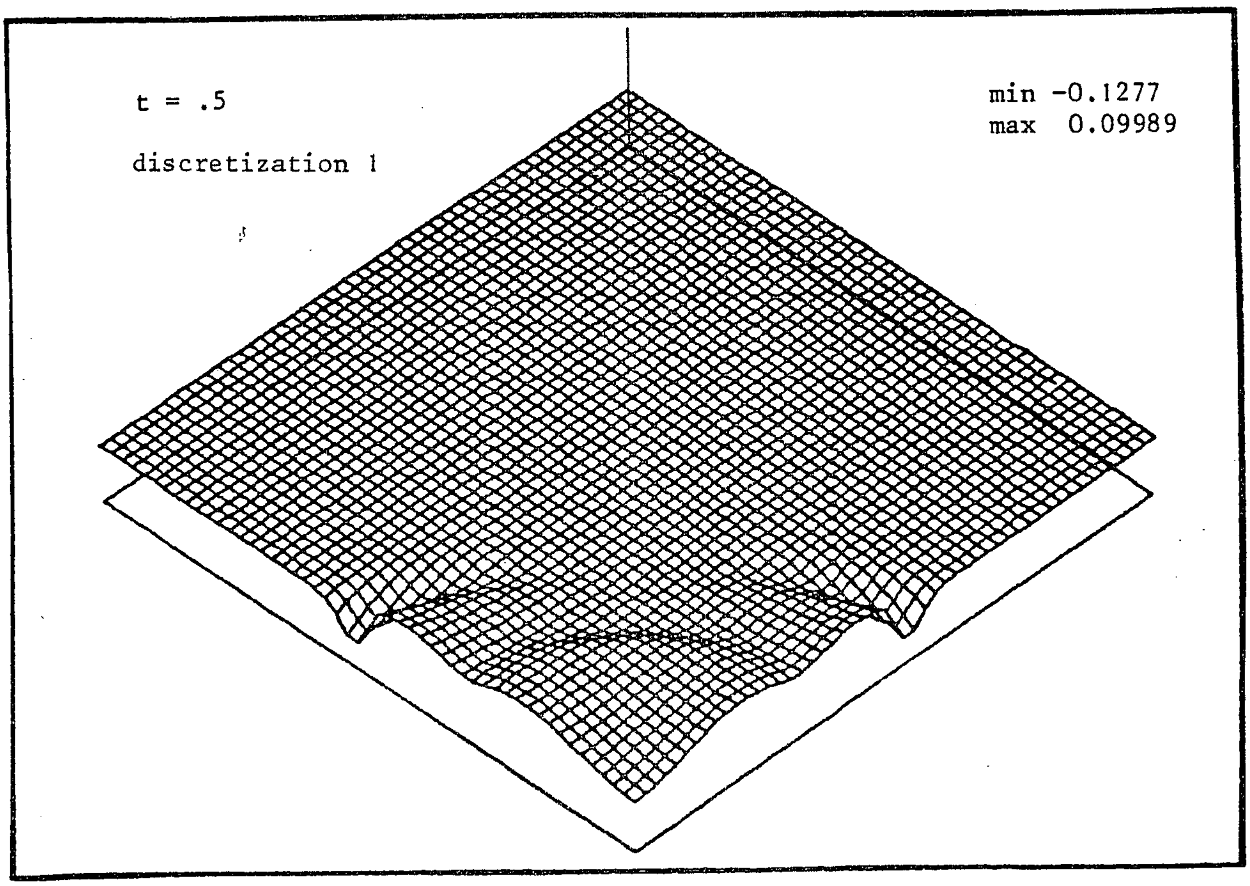


Figure 38 : difference between solution calculated with γ=0.1 and exact solution

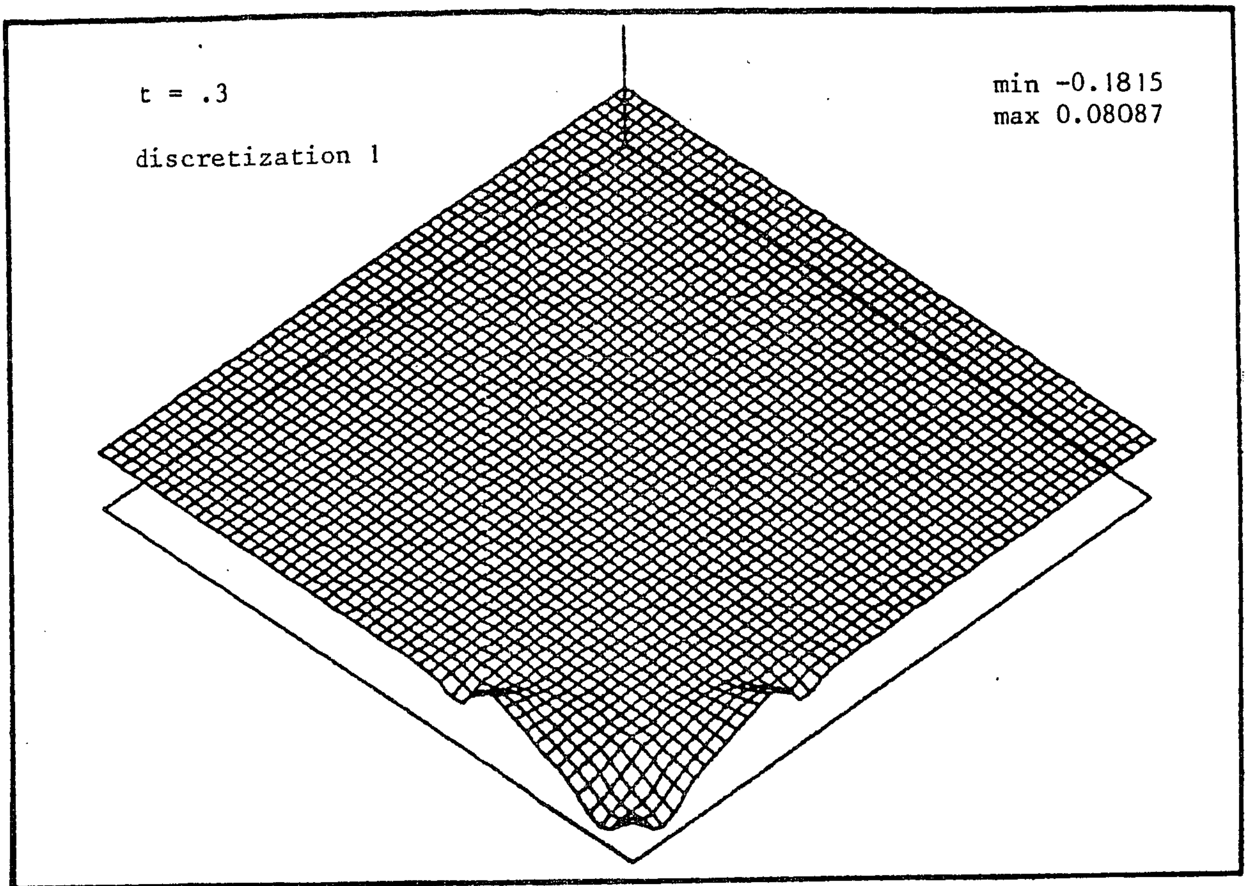


Figure 39 : difference between solution calculated with γ=3. and exact solution

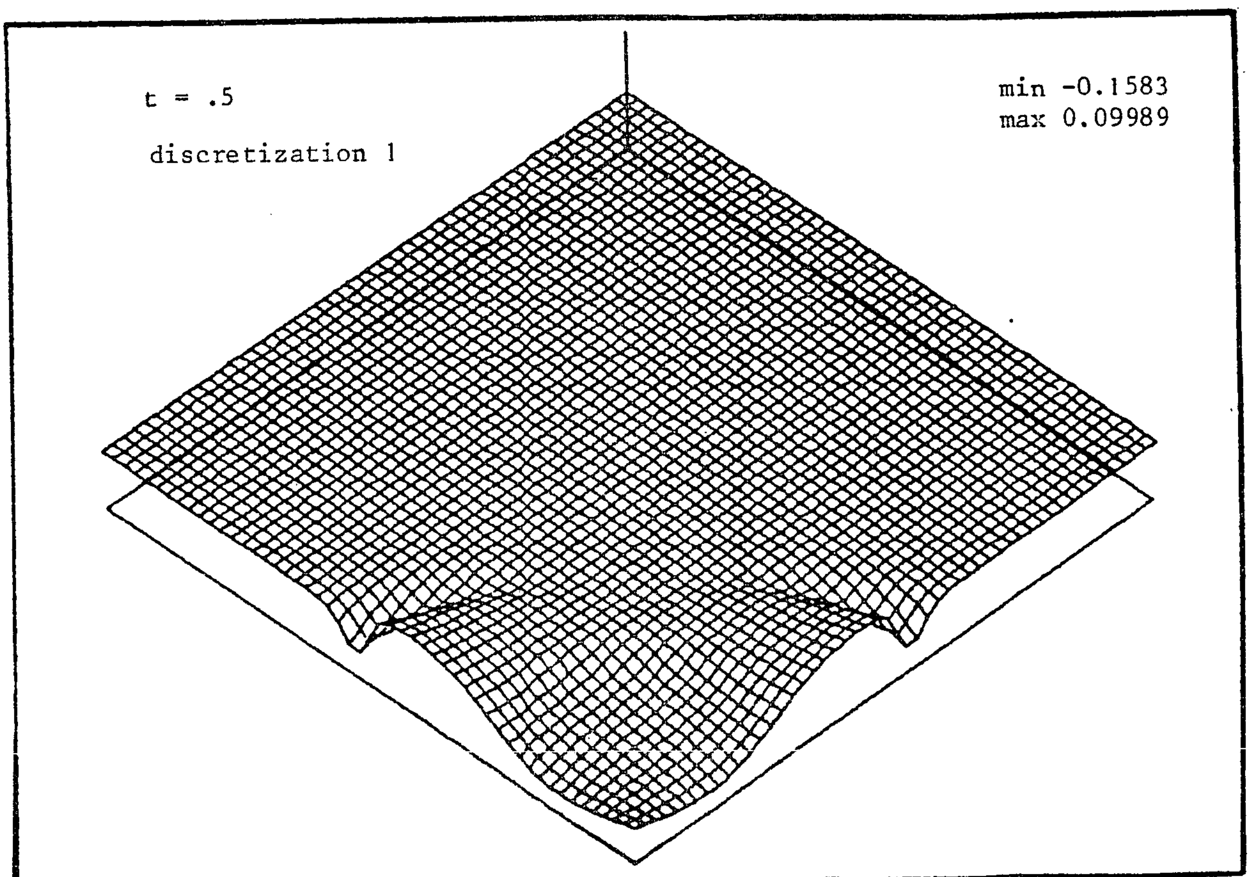


Figure 40 : difference between solution calculated with γ=3. and exact solution

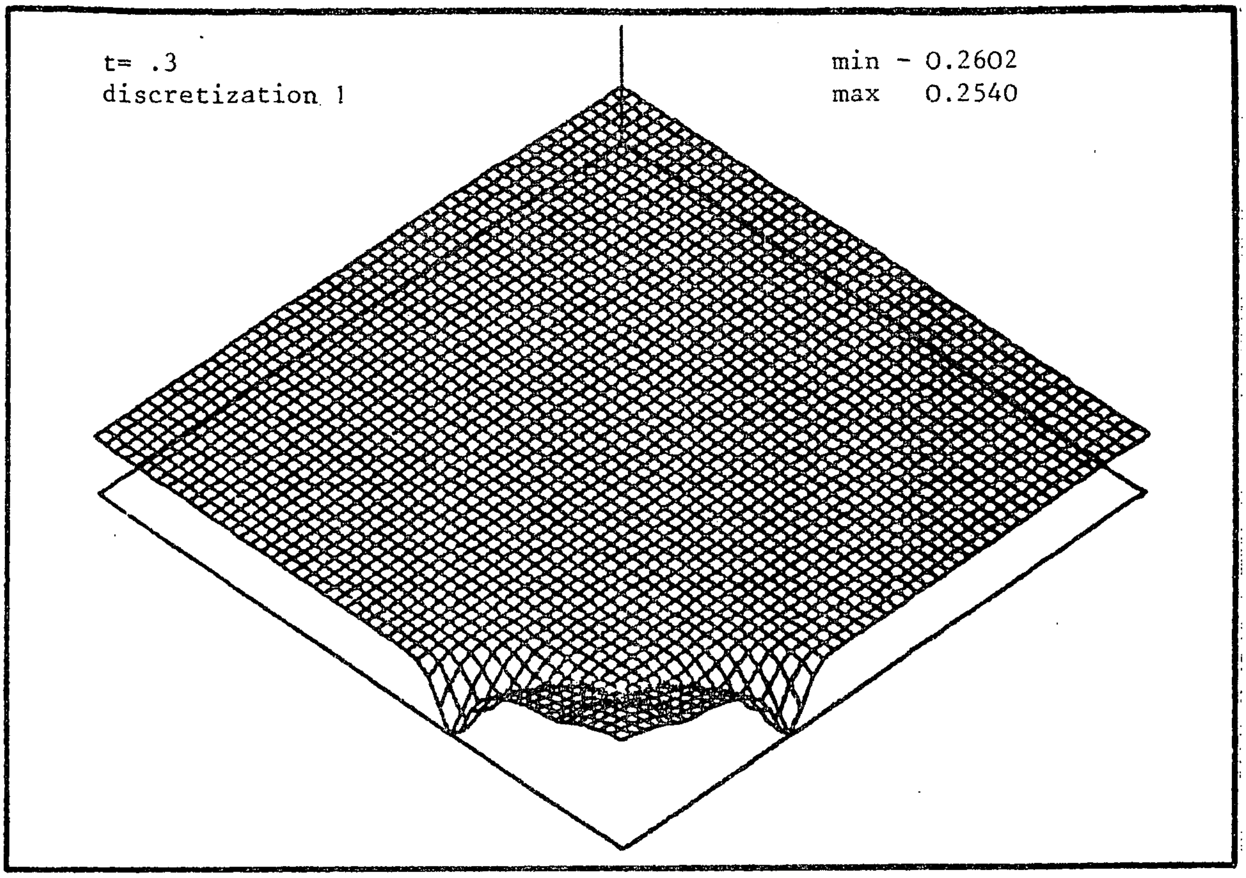


Figure 41: Difference between solution calculated with 1st order ABC and exact solution.

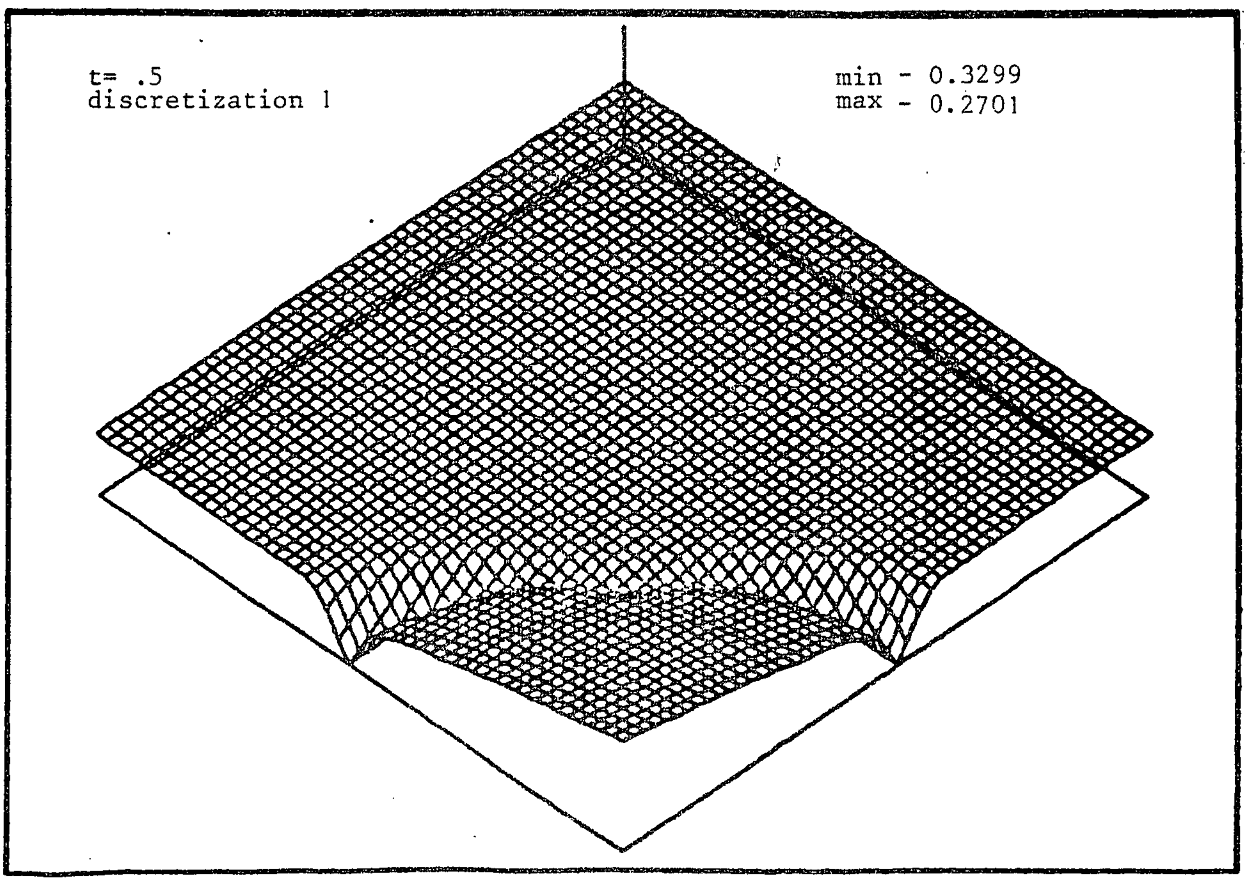


Figure 42: Difference between solution calculated with 1st order ABC and exact solution.

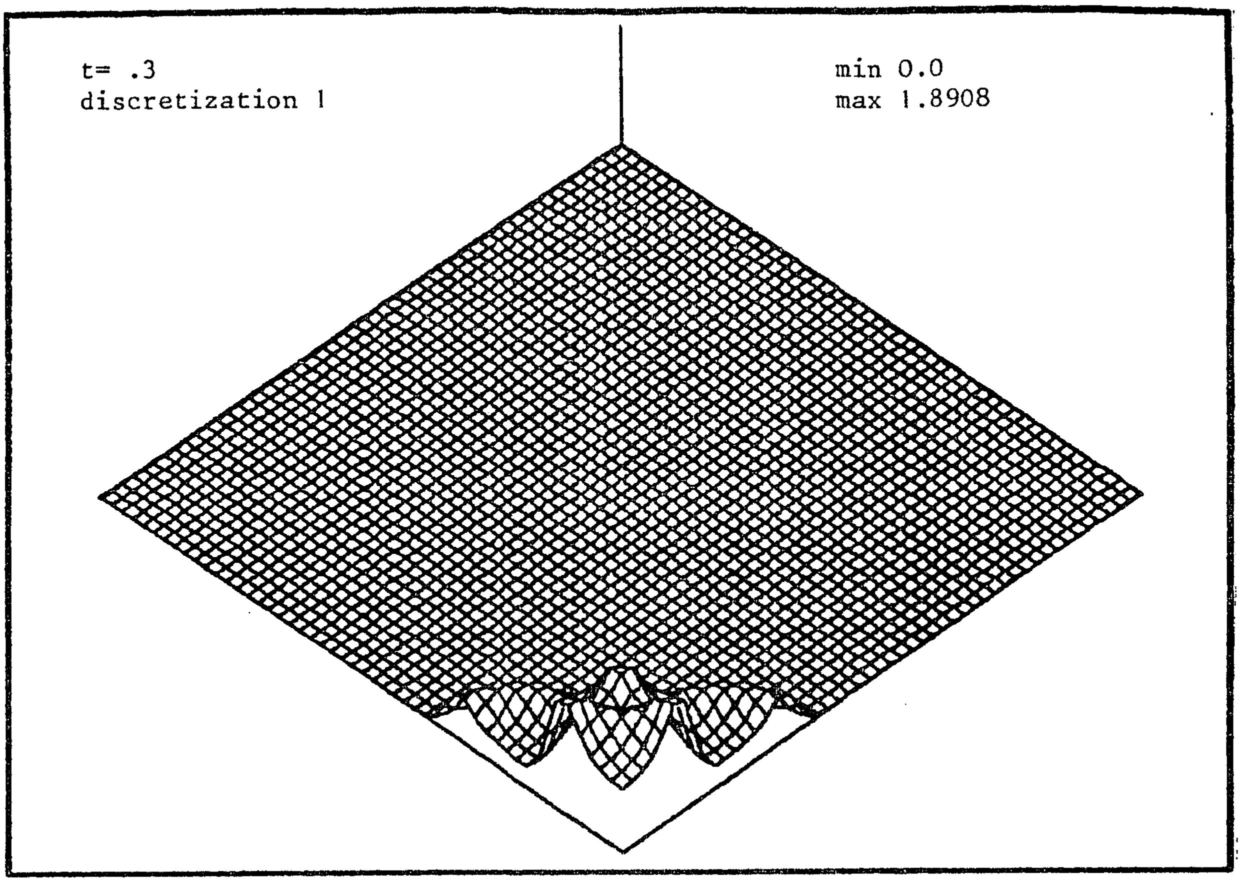


Figure 43: Difference between solution calculated with Neumann BC and exact solution

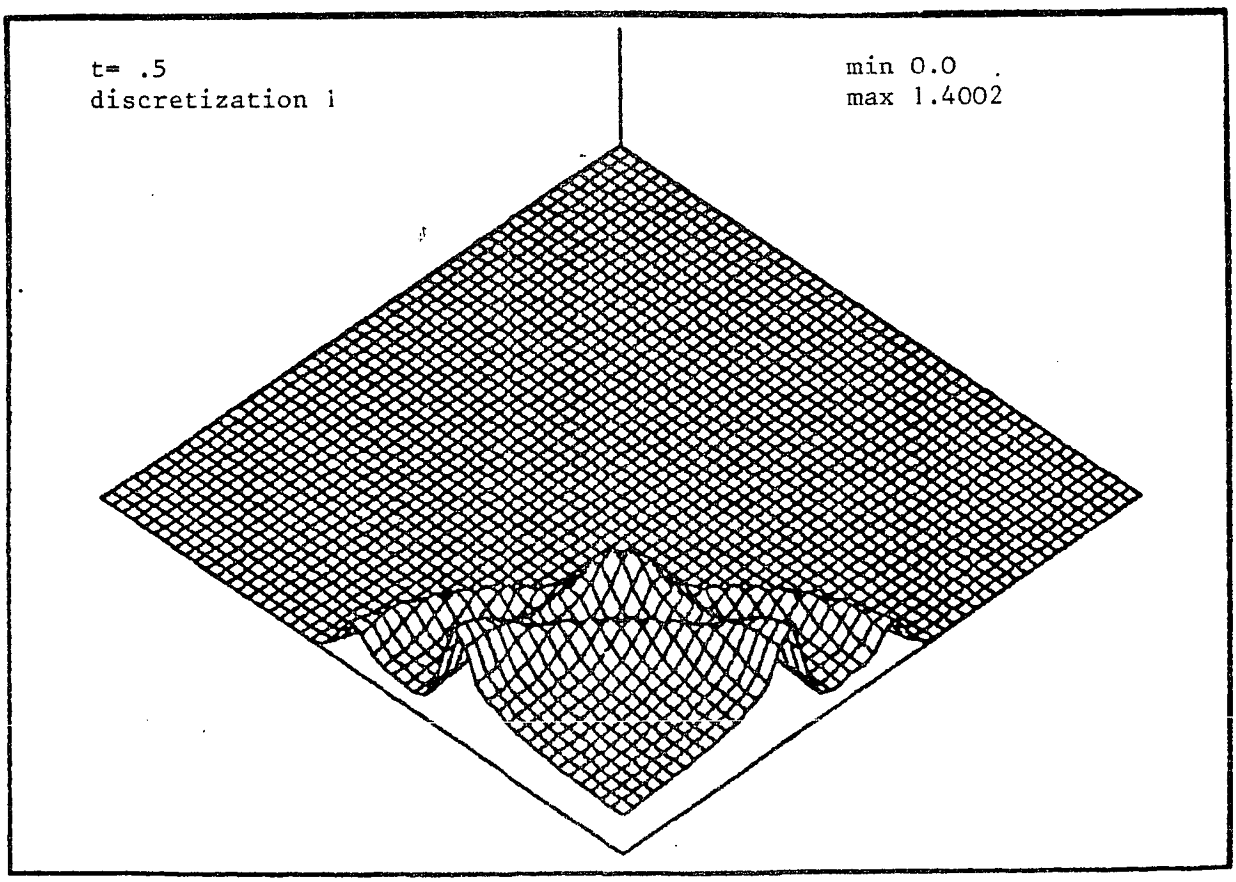


Figure 44: Difference between solution calculated with Neumann BC and exact solution

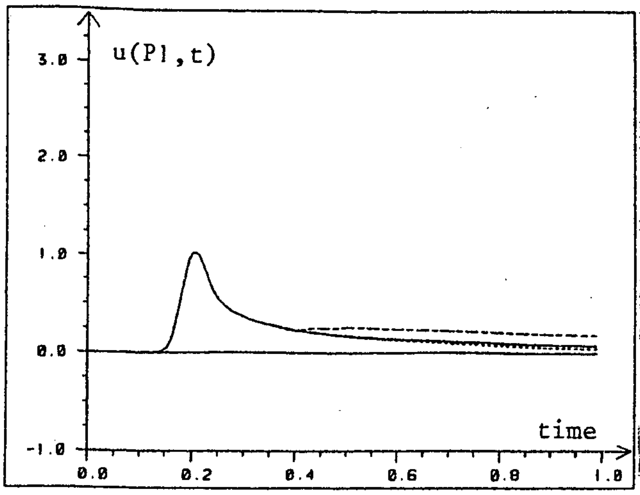


Figure 45: Response at P1=(.79,.79)
exact solution
2nd order ABC (γ=1.5)
1st order ABC

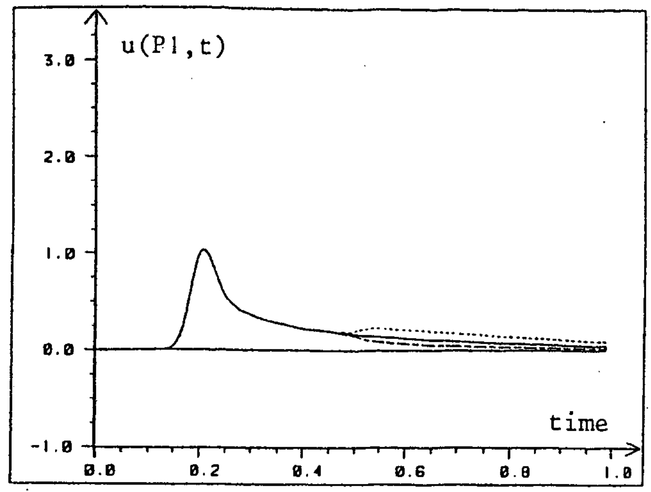


Figure 46: Response at P1=(.79,.79)
2nd order ABC γ=1.5
2nd order ABC γ=0.1
2nd order ABC γ=3.0

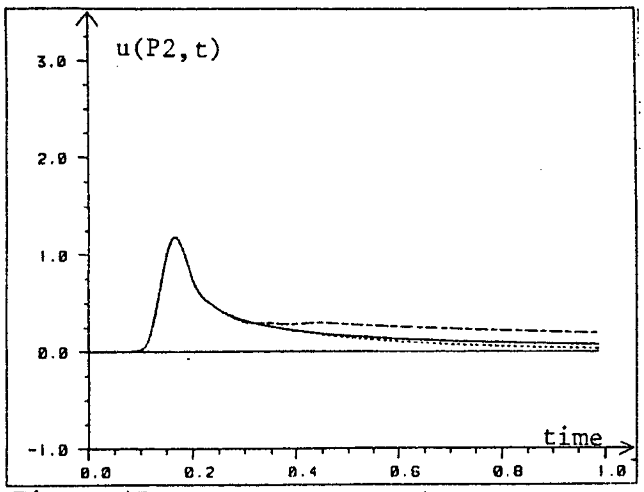


Figure 47: Response at P2=(.89,.79)
exact solution
2nd order ABC (γ=1.5)
1st order ABC

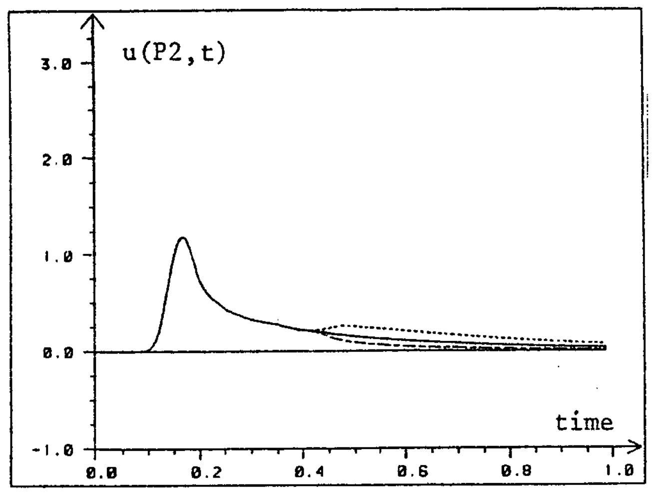


Figure 48: Response at P2=(.89,.79)
2nd order ABC γ=1.5
2nd order ABC γ=0.1
2nd order ABC γ=3.0

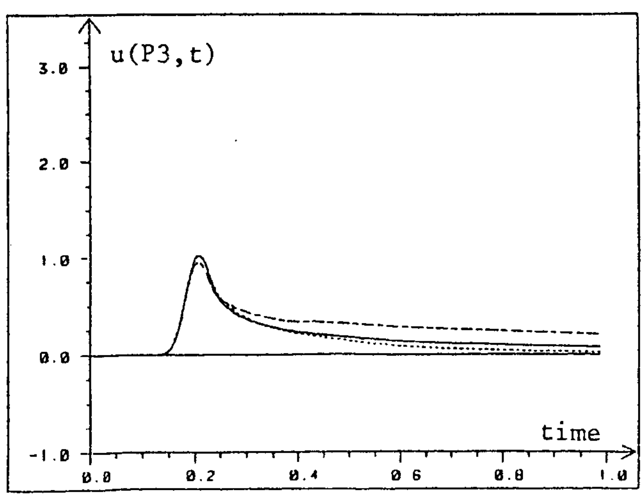


Figure 49: Response at P3=(.99,.79)
exact solution
2nd order ABC (γ=1.5)
1st order ABC

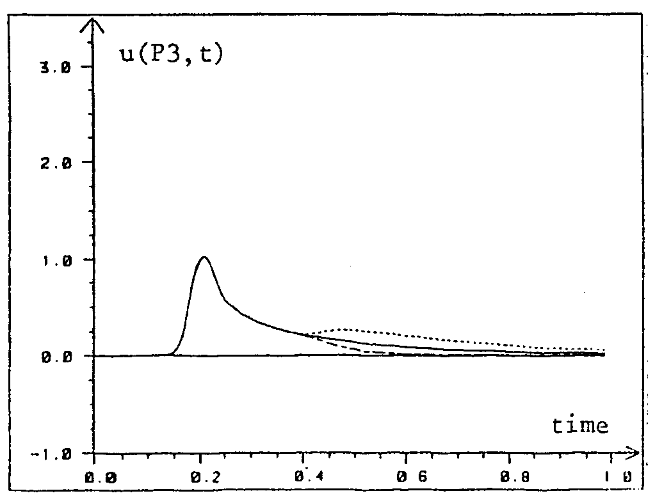


Figure 50: Response at P3=(.99,.79)
2nd order ABC γ=1.5
2nd order ABC γ=0.1
2nd order ABC γ=3.0

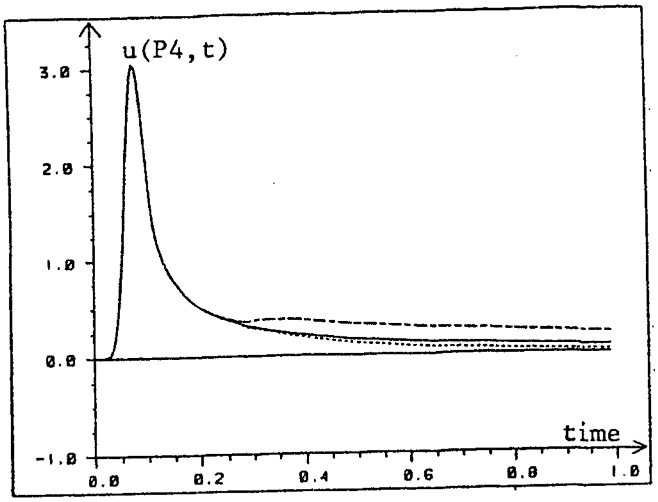


Figure 51: Response at P4=(.89,.89)
exact solution
2nd order ABC (γ=1.5)
1st order ABC

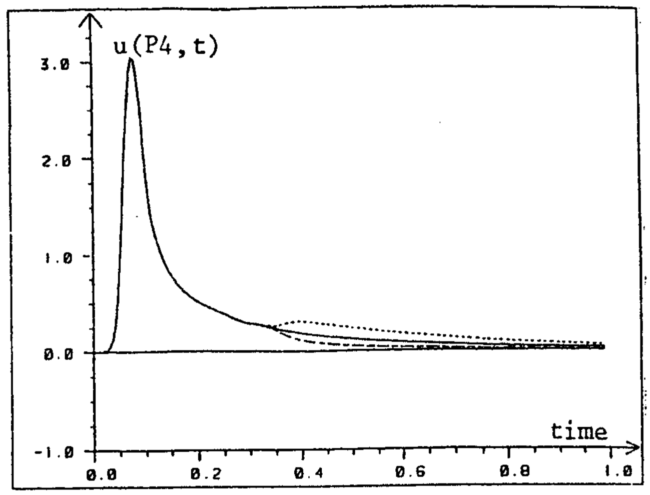


Figure 52: Response at P4=(.89,.89)
2nd order ABC γ=1.5
2nd order ABC γ=0.1
2nd order ABC γ=3.0

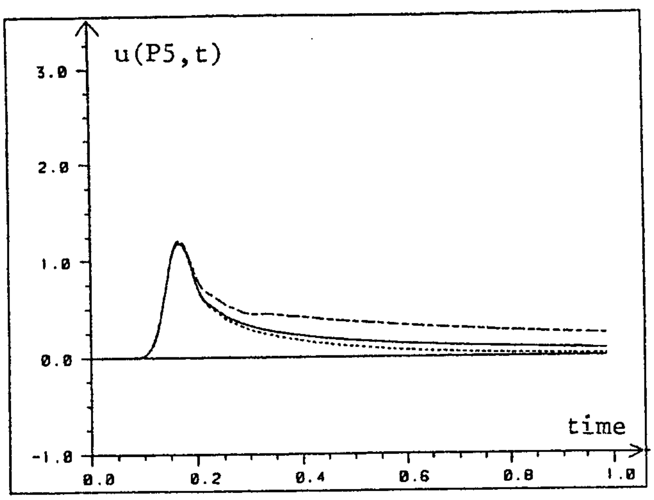


Figure 53: Response at P5=(.99,.89)
exact solution
2nd order ABC (γ=1.5)
1st order ABC

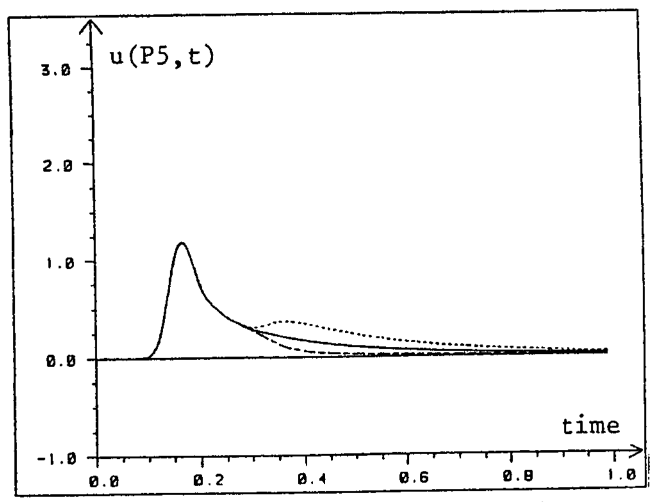


Figure 54: Response at P5=(.99,.89)
2nd order ABC γ=1.5
2nd order ABC γ=0.1
2nd order ABC γ=3.0

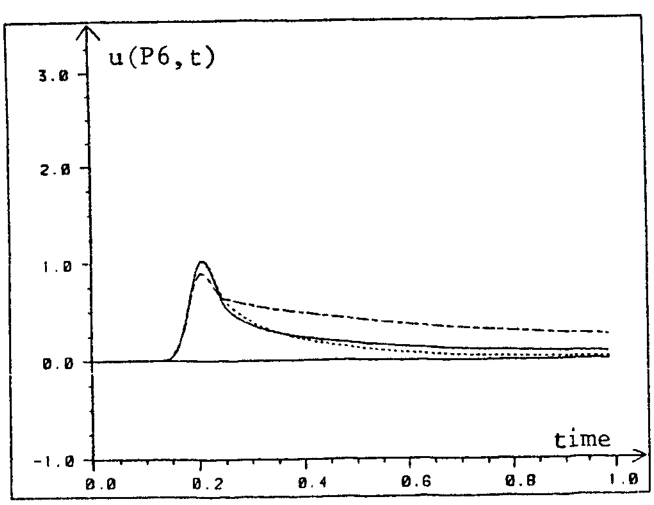


Figure 55: Response at P6=(.99,.99)
exact solution
2nd order ABC (γ=1.5)
1st order ABC

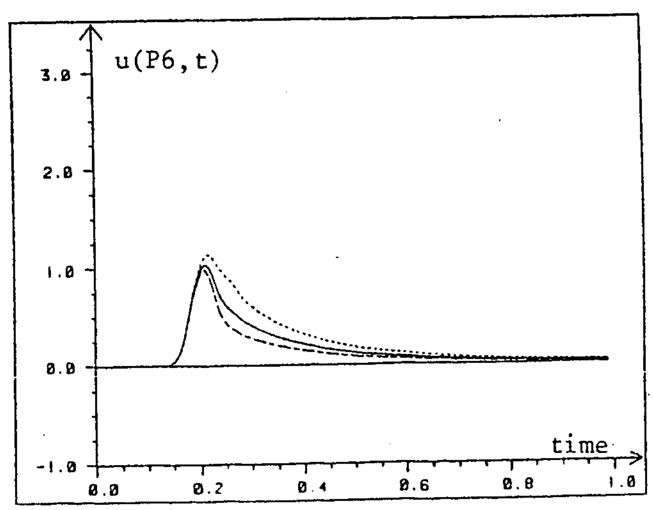


Figure 56: Response at P6=(.99,.99)
2nd order ABC γ=1.5
2nd order ABC γ=0.1
2nd order ABC γ=3.0

REFERENCES

[1] **BAMBERGER A., JOLY P., ROBERTS J.E.,** *"Second order absorbing boundary conditions for the wave equation : a solution for the corner problem"* submitted for publication.

[2] **COURANT R., HILBERT D.,** *"Methods of Mathematical Physics"*, Vol. 2, Interscience Publishers, New York (1962).

[3] **DAUTRAY R., LIONS J.L.,** *"Analyse Mathématique et Calcul Numérique pour les sciences et les techniques"*, Masson (1985)

[4] **ENGQUIST B., MAJDA A.,** *"Absorbing boundary conditions for the numerical simulation of waves"* Math of Comp., Vol. 31, n° 139, pp. 629-651 (1977).

[5] **ENGQUIST B., MAJDA A.,** *"Radiation Boundary Conditions for Acoustic and Elastic Wave Calculations"*, Comm. Pure Appl. Math, Vof. 32, pp. 313-357, (1979).

[6] **KREISS H.O.,** *"Initial boundary value problems for hyperbolic systems"*, Comm. Pure Appl. Math., Vol. 23, pp. 277-298 (1970).

[7] **KUPKA I.A.K., OSHER S.J.,** *"On the wave equation in a multi-dimensional corner"*, Comm. Pure Appl. Math., Vol. 24, pp. 381-393, (1979).

[8] **OSHER S.,** *"Initial boundary value problems for hyperbolic systems in regions with corners"*, Trans. Amer. Math. Soc., Vol 176, pp. 141-164, (1973).

[9] **RAVIART P.A., J.M. THOMAS.,** *"Introduction à l'Analyse Numérique des Equations aux Dérivées Partielles*, Masson, Paris, (1983).

[10] **RICHTMYER R.D., MORTON K.W,** *"Difference Methods for Initial Value Problems"*, Interscience Publishers, New York (1967).

[11] **SARASON L., SMOLLER J.A.,** *"Geometrical optics and corner problem"*,

****